# Salem sets and restriction properties of Fourier transforms

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## 0.1 Introduction

The main purpose of this paper is to give an analogue on the real line of restriction phenomena of Fourier transforms first discovered by E. M. Stein in higher dimensions in the late sixties. In fact, we are going to establish a result on the real line which is almost as sharp as the Stein-Tomas theorem. This theorem says, for a function from the Lebesgue space $L^p(\mathbf{R}^n), 1 \leq p \leq 2(n+1)/(n+3), n > 1$, that its Fourier transforms restricted to the unit sphere $S^{n-1}$ in $n$-dimensional Euclidean space is well defined and square integrable against the uniform measure on $S^{n-1}$. Starting in the seventies there have been many generalizations of this result, mainly for situations where the unit sphere is replaced by some smooth submanifold of $\mathbf{R}^n$ satisfying suitable curvature conditions, but similar questions have also been discussed in the case where the Fourier transform is replaced by a more general oscillatory integral operator (see e.g. [21, 6, 7, 18, 20, 16]). More recently J. Bourgain[2] (see also, [3, 22]) has developed a method to improve on the Stein-Tomas result for $n \geq 3$ –the case $n = 2$ was settled by Stein in [9]. The literature on the subject might suggest that these restriction phenomena are genuinely $n$-dimensional aspects of Fourier Analysis (see, e.g.[10]). We will see however, that a proper analogue of restriction phenomena for Fourier transforms can be developed on the real line. In part this point of view is motivated by a recent result of J. Bourgain[4, 23] showing that the bounds conjectured by H. Montgomery[17, p. 142] on finite Dirichlet sums imply that Kakeya sets, i.e. sets containing a line segment in every direction in $\mathbf{R}^n$, have Hausdorff dimension $n$. This conclusion is known to follow in a more natural way from the conjectured optimal restriction estimates for spheres, which in dual form state that

$$\int_{\mathbf{R}^n} |\widehat{fd\sigma}|^p \; dx \leq C \; \|f\|^p_{L^q(S^{n-1},d\sigma)}, \tag{1}$$

for $p > 2n/(n-1), q = \infty$ and $d\sigma$ being the uniform measure on $S^{n-1}$.

The question we ask here is whether we can replace, in (1), the measure $d\sigma$ by a probability measure $d\mu$ supported on a fractional dimensional compact set $E$ on $\mathbf{R}$ and still obtain the estimate (1) with appropriate nontrivial exponents $p, q$. This, of course, requires that the Fourier transform of a measure $d\mu$ supported on a set $E$ of Hausdorff dimension $\alpha \in (0,1)$ lies in $L^p(\mathbf{R})$ for some $p < \infty$. It is known that this implies strong conditions on the structure of the set $E$ and $p > 2/\alpha$. Now, by answering a question of A. Buerling, R. Salem in [19] constructed for a fixed $\alpha \in (0,1)$ and each $\epsilon > 0$ measures $d\mu$ supported on a set of Hausdorff dimension $\alpha$ whose Fourier transform lies in $L^p(\mathbf{R})$ for $p > 2/\alpha + \epsilon$ (see [12] for a nice historical background). For extensions of Salem's result see [11, 13].

It seems natural to conjecture for the measures constructed by Salem that the following $(L^q, L^p)$-estimate holds:

$$\int_{\mathbf{R}} |\widehat{fd\mu}|^p \; dx \leq C \; \|f\|^p_{L^q(d\mu)}, \tag{2}$$

with $p > 2/\alpha + \epsilon$ and $q = \infty$. By multiplying the measure $\mu$ with a nonnegative weight function a factorization argument would allow us to lower the $q$-exponent.

An interpolation argument with the trivial $(L^1, L^\infty)$-estimate would then give an $(L^2, L^p)$-estimate which is essentially what we are going to proof here for the original measure $d\mu$.

We would like to point out that we cannot exclude the possibility that the inequality in (2) does hold for all $p > \frac{2}{\alpha} + \epsilon$ and $q = 2$. Note that in case of the sphere the sharpness of the Stein-Tomas result follows from the curvature of the sphere, or, by considering the equivalent problem for the $n$-torus from the fact that $\mathbf{Z}^n \cap \{R < |x| \leq R+1\}$ contains an $n-1$-dimensional arithmetic progression of size $R^{(n-1)/2}$. The analogue of this *curvature condition* for a $1/R$ neighborhood of the set $E = supp(\mu)$ would be a suitable estimate on the size of an arithmetic progression contained in it.

The importance of restriction theorems in analysis lies in their use to exploit certain cancellation properties of convolution operators. For example, the Stein-Tomas result is central in understanding spherical summation operators corresponding to the Bochner-Riesz multipliers. The restriction theorems we are going to show here will allows us to construct new $L^p(\mathbf{R})$-multipliers which in a sense play the same role that the Bochner-Riesz multipliers do in $\mathbf{R}^n$ for $n > 1$.

As a further application we will see that the algebras $M_p(I) \subset L^\infty(\mathbf{R})$ of bounded convolution operators on $L^p(\mathbf{R})$ whose corresponding multipliers are supported in an interval $I$ have the property that $M_p(I) \neq M_q(I)$ for $1 \leq p < q < 2$. The fact that the $M_p's$ are separated by characteristic functions has been shown in [15] using Bourgain's solution of the $\Lambda_p$-problem.

## 0.2 Hausdorff and Fourier dimension

We will begin with a short review of two notions of dimension. For details we refer to Kahane's book [11]. By a theorem of Frostman it is known that if $E \subset \mathbf{R}^n$ is a compact set of Hausdorff dimension $\alpha$, then there is a probability measure $\mu$ supported on $E$ satisfying $\mu(B_r(x)) \leq C\, r^\alpha$, where $B_r(x)$ denotes a ball of radius $r$ centered at $x$.

Therefore, the $\beta$-potential $(0 < \beta < n)$ of $\mu$ at a point $x$ defined as

$$I_\beta(\mu)(x) = \int \frac{d\mu(y)}{|x-y|^\beta}$$

is a bounded function as long as $\beta < \alpha$ and this implies the finiteness of the $\beta$-energy of $\mu$ defined as

$$I_\beta(\mu) = \int\int \frac{d\mu(y) d\mu(x)}{|x-y|^\beta}$$

for $\beta < \alpha$. On the other hand the theorem of Frostman shows that if $I_\alpha(\nu) < \infty$, for some probability measure $\nu$ supported on a compact set $E$, then its Hausdorff dimension $dim_H E \geq \alpha$. Since the Fourier transform of $|x|^{-\alpha}, 0 < \alpha < n$, is $C\, |x|^{\alpha-n}$, one can show (see [5]):

$$I_\alpha(\mu) = c \int_{\mathbf{R}^n} \frac{|\widehat{d\mu}(y)|^2}{|y|^{n-\alpha}}\, dy.$$

Thus $I_\alpha(\mu) < \infty$ provides some information on the size of $\widehat{d\mu}$, although it does not even imply that $\widehat{d\mu}(x) \to 0$ as $x \to \infty$ (consider e.g. $E = [0,1] \subset \mathbf{R}^2$).

We define the Fourier dimension of a compact set $E \subset \mathbf{R}^n$, denoted by $\dim_F E$, as the supremum of $\beta \geq 0$ such that for some probability measure $d\mu$ supported on $E$

$$|\widehat{d\mu}(x)| \leq C|x|^{-\frac{\beta}{2}}.$$

Since the last inequality –or even the weaker assumption that $\widehat{d\mu} \in L^p(\mathbf{R}^n)$ for $p > 2n/\beta$– implies $I_\alpha(\mu) < \infty$ for $\alpha < \beta$, we always have $\dim_F E \leq \dim_H E$. In case $E$ is a compact smooth $\alpha$-dimensional submanifold of $\mathbf{R}^n$ and $d\mu$ is the measure induced by Lebesgue measure on $\mathbf{R}^n$ we may expect an isotropic decay of the form $|x|^{-\alpha/2}$ only under some conditions on the curvature and on the dimension of $E$. Here are some examples:

(1) The unit sphere in $\mathbf{R}^n$ has Fourier dimension $n-1$.

(2) The boundary of the unit cube in $\mathbf{R}^n$ has Fourier dimension 0.

(3) The Cantor middle third set has Fourier dimension 0 and Hausdorff dimension $\log 2/\log 3$.

(4) R. Kaufman [13] has shown that for $t > 0$ the set $E_t$ of those real numbers $x \in [0,1]$ such that

$$||qx|| \leq q^{-1-t}$$

has solutions for arbitrarily large integers $q$, $||x||$ denotes the distance to the nearest integer, has Fourier dimension and Hausdorff dimension equal to $\frac{2}{2+t}$ (see also [1]). The statement about the Hausdorff dimension for $E_t$ was shown by V. Jarnik (see [8]).

These examples show that Hausdorff dimension and Fourier dimension do not agree in general. This is not surprising since Hausdorff dimension measures a metric property of a set, whereas the Fourier dimension measures an arithmetic property of a set. However, the sets in examples (1) and (4) do have the property that their Fourier dimension and Hausdorff dimension agree. Before R. Kaufman showed (by a deterministic method) that the sets in example (4) share this property, there were probabilistic constructions which provided subsets of the real line of fractional Hausdorff dimension having the same Fourier dimension. In fact, the existence of sets having this property was first shown by R. Salem [19] and they are named after him. Later J.-P. Kahane [11] provided a rich class of Salem sets by showing that images of compact sets of a given Hausdorff dimension under Brownian motion are almost surely Salem sets.

## 0.3 Salem's construction

We begin with a generalization of the classical Cantor type construction (see [24, p.194]). Let $M > 2$ be an even integer, $N = M^M$. Choose $\xi \in (0, 1/N)$ and let points $0 < a_1 < a_2 < \cdots < a_N < 1$ be given such that they are linearly independent over the rational numbers and such that

$$0 < a_1 < 1/N - \xi \qquad \text{and} \qquad \xi < a_k - a_{k-1} < 1/N, \quad k = 2, \ldots, N. \tag{3}$$

On an interval $[A, B]$ of length $L$, we perform a dissection of type $(N, a_1, \ldots, a_N, \xi)$ by calling the closed intervals

$$[A + La_k, A + L(a_k + \xi)], \quad k = 1, \ldots, N$$

*white* and the complementary intervals *black*. Let $\Xi = (\xi_k)_{k \geq 1}$ be a sequence such that

$$(1 - \frac{1}{2k^2})\ \xi \leq \xi_k \leq \xi, \qquad k \geq 1.$$

Starting with $E_0 = [0, 1]$, we perform a dissection of type $(N, a_1, ..., a_N, \xi_1)$ and remove the black intervals, thereby obtaining a set $E_1$ which is a union of $N$ intervals each of length $\xi_1$. On each remaining interval we perform a dissection of type $(N, a_1, ..., a_N, \xi_2)$, remove the black intervals and so obtain a set $E_2$ of $N^2$ intervals each of length $\xi_1\xi_2$. After $n$ steps we obtain a set $E_n$ of $N^n$ intervals each of length $\xi_1\xi_2...\xi_n$. Put

$$E = \bigcap_{n \geq 0} E_n.$$

Then $E$ is a perfect set having measure 0 if $N^n\xi^n \to 0$ and Hausdorff dimension $\alpha$ if we choose $\xi = N^{-1/\alpha}$.

For each $n \in \mathbf{N}$, let $F_n$ be a continuous nondecreasing function satisfying:

- $F_n(0) = 0$ for $x \leq 0$ and $F_n(1) = 1$ for $x \geq 1$.
- $F_n$ increases linearly by $1/N^n$ on each white interval in $E_n$.
- $F_n$ is constant on each black interval in $E_n$.

Let $F = \lim_{n \to \infty} F_n$. Then $F$ is a nondecreasing continuous function. To compute the Fourier transform of the corresponding measure, we put $P(x) = \frac{1}{N} \sum_{1 \leq k \leq N}\ e^{ia_k x}$. It is easy to see that

$$\widehat{dF_{n+1}}(x) = P(x)P(x\xi_1)...P(x\xi_1...\xi_n)\ \frac{e^{i\xi_1 \cdots \xi_n x} - 1}{i\xi_1...\xi_n x}.$$

Hence

$$\widehat{dF}(x) = P(x) \prod_{n \geq 1} P(x\xi_1...\xi_n).$$

Note that $E$ and $F$ do depend on $\Xi$. The mentioned result of Salem [19] is the following one

**Theorem 0.4.** *Given $\epsilon > 0$, there is $M > 2$ and a sequence $\Xi$ as above such that*

$$|\widehat{dF}(x)| \leq C_\epsilon\ |x|^{-\frac{\alpha}{2} + \epsilon}.$$

To make the paper self contained we will sketch the proof below. We first need to get bounds on each individual factor in the product forming $\widehat{dF}$.

**Lemma 0.5.** (i) $|P(x)| \leq 1$.

(ii) *Let $p = 2k$ be an even integer. Then there exists $T_0 = T_0(N, a_i, p)$ such that for $T > T_0$ and all $b \in \mathbf{R}$*

$$\left( \frac{1}{T} \int_b^{b+T} |P(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{\sqrt{p}}{\sqrt{N}}.$$

The first part is trivial and for the second we apply to the $k$-th powers of $P$ the inequality

$$\frac{1}{T} \int_b^{b+T} |\sum_\gamma c_\gamma e^{i\gamma x}|^2 dx \leq \sum |c_\gamma|^2 + \sum_{\gamma \neq \gamma'} \frac{|c_\gamma c_{\gamma'}|}{T|\gamma - \gamma'|},$$

where the frequencies $\gamma$ lie in a finite set of real numbers. Choosing $T$ large enough and using the linear independence of the $a_k$'s we may bound the term $T|\gamma - \gamma'|$ from below. Then a simple computation gives the bound in the lemma.

Now the random construction comes in. Let $\zeta_n, n \geq 1$, be a Steinhaus sequence, i.e. a sequence of independent random variables mapping $[0,1]$ onto the unit cube in infinite dimensions (see [11]). It is known that for measurable functions $f$ on $\mathbf{R}^K$ we have

$$\int_0^1 f(\zeta_1(\omega), \ldots, \zeta_K(\omega)) \, d\omega = \int_{[0,1]^K} f(t_1, \ldots, t_K) \, dt_1 \ldots dt_K.$$

We put $A_n = (1 - \frac{1}{2n^2})\xi$ and let $\xi_n(\omega) = A_n + (\xi - A_n)\zeta_n(\omega)$. Since $|P(x)| \leq 1$ we get

$$\begin{aligned} \int_0^1 |\widehat{dF_\omega}(x)|^M d\omega &\leq \int_0^1 \prod_{K \geq n \geq 1} |P(x\xi_1(\omega)...\xi_n(\omega))|^M d\omega \\ &= \int_{[0,1]^K} \prod_{K \geq n \geq 1} |P(x\xi_1...\xi_n)|^M \, d\zeta_1 \ldots d\zeta_K. \end{aligned}$$

Integrating the last factor in the product with respect to the last variable $\zeta_K$ gives

$$\int_{[0,1]} |P(x\xi_1...\xi_K)|^M \, d\zeta_K = \frac{1}{T} \int_b^{b+T} |P(x)|^M dx,$$

where $T = x\xi_1 \ldots \xi_{K-1}(\xi - A_K) \geq C \, x \, \xi^K \, K^{-2}$. This last expression is monotone decreasing in $K$. Hence, choosing $K = K(x)$ such that $C \, x \xi^K \, K^{-2} \geq T_0$ we may apply Lemma 0.3.2. Moreover, having chosen $K$ we also have, for $n < K$, that $C\,x\,\xi^n\,n^{-2} \geq T_0$. Hence, successive integration yields

$$\begin{aligned}
\int_0^1 |\widehat{dF_\omega}(x)|^M d\omega &\leq \int_{[0,1]^K} \prod_{K\geq n\geq 1} |P(x\xi_1...\xi_n)|^M \; d\zeta_1 \ldots d\zeta_K \\
&\leq \left(\frac{\sqrt{M}}{\sqrt{N}}\right)^M \int_{[0,1]^{K-1}} \prod_{K-1\geq n\geq 1} |P(x\xi_1...\xi_n)|^M \; d\zeta_1 \ldots d\zeta_{K-1} \\
&\leq \left(\frac{\sqrt{M}}{\sqrt{N}}\right)^{MK}.
\end{aligned}$$

A simple computation shows, for $x$ sufficiently large, that we can take

$$K(x) > (1 - \frac{1}{\sqrt{M}})\frac{\log x}{|\log \xi|}.$$

Hence, for $|x| > C(T_0, \xi)$ and using $\alpha = \log N / |\log \xi|$, we obtain

$$\int_0^1 |\widehat{dF_\omega}(x)|^M d\omega \leq C\; |x|^{-\frac{\alpha}{2}M(1-\frac{1}{\sqrt{M}})(1-\frac{\log M}{\log N})} \leq C\; |x|^{-\frac{\alpha}{2}M(1-\frac{2}{\sqrt{M}})},$$

where in the last line we have put $N = M^M$. This gives

$$\int_0^1 \sum_{n\geq 1} \left(|n|^{\frac{\alpha}{2}(1-\frac{4}{\sqrt{M}})}|\widehat{dF_\omega}(\epsilon n)|\right)^M \; d\omega \leq C,$$

uniformly for $\epsilon \in [1, 2]$. Therefore, the series converges for almost every $\omega$ which in turn implies

$$|\widehat{dF_\omega}(x)| \leq C_M\; |x|^{-\frac{\alpha}{2}(1-\frac{4}{\sqrt{M}})}, \qquad \text{for a.e. } \omega \in [0,1]. \tag{4}$$

Besides Salem's result we will need the following regularity property of the function $F = F_\omega$ which holds for all $\omega \in [0, 1]$.

**Proposition 0.6.** *There is a constant $C$, depending only on $N$ such that for $x, y \in$* **R**

$$|F(x) - F(y)| \leq C\; |x - y|^\alpha.$$

To the prove this let $x, y \in [0, 1]$ and suppose that $y > x$. Since $F$ is constant on black intervals complementary to $E$, we may assume that $x, y$ lie in $E$. Let $k$ be the smallest integer such that after $k$ dissections at least two black intervals lie in $[x, y]$. Then $[x, y]$ contains a white interval and

$$y - x \geq \xi_1 \ldots \xi_k \geq \xi^k \prod_{1\leq m\leq k} (1 - \frac{1}{2m^2}) \geq C\; \xi^k.$$

Now, after $k-1$ dissections there is at most one black interval $(a,b)$ contained in $[x,y]$. Hence using $N=\frac{1}{\xi^\alpha}$ we find

$$F(y)-F(x)=F(y)-F(b)+F(a)-F(x)\le \frac{2}{N^{k-1}}\le C\ \xi^{k\alpha}\le C\ (y-x)^\alpha.$$

**Remark 0.7.** *In [19] R. Salem provides a refined method to construct a monotone function $F$ whose Fourier transform is of order $\Omega(x)|x|^{-\alpha/2}$, with $\Omega$ increasing slower then any power of $x$. He achieves this by increasing, at each step of the dissection, the number of intervals, i.e. $E_k$ is obtained from $E_{k-1}$ by performing a dissection of type $(N(k),a_1^{(k)},\dots,a_{N(k)}^{(k)},\xi_k)$ on each interval in $E_{k-1}$ with $N(k)$ increasing; in fact, we may take $N(k)=k+1$. Also, for fixed $k=1,2,..$ the $\xi_k$ and $a_j^{(n)}$ satisfy the conditions in (3) and $\xi^{(n)}(1-1/(n+1)^2)\le\xi_n\le\xi^{(n)}$ with $\xi^{(n)}=N(n)^{-1/\alpha}$. In this situation a slight variation of the argument proving Proposition 0.3.3 gives for the analogous function $F$:*

$$F(y)-F(x)\le\frac{2}{N(1)\dots N(k-1)}=2N(k)(\xi^{(1)}\dots\xi^{(k)})^\alpha.$$

*Now, with the particular choice $N(k)=k+1$ we find the last term is bounded by a constant times $(k+1)(\log(k+1)!)^{-1}\ (y-x)^\alpha\ (1+|\log(y-x)|)$. Hence, Stirling's formula gives*

$$|F(y)-F(x)|\le C\ |y-x|^\alpha\ (1+|\log(y-x)|).$$

## 0.8 A restriction theorem on the real line

As an application of Young's inequality and the fact that the Fourier transform of the measures constructed by Salem lie in a nontrivial $L^p(\mathbf{R})$, one can get a restriction result for Fourier transforms of functions in $L^p(\mathbf{R})$ for $p$ close to 1 (by following E.M. Stein's original argument [9]). To improve on this range we will prove the following result

**Theorem 0.9.** *Let $\mu$ be a compactly supported positive measure on $\mathbf{R}^n$ which satisfies the following properties.*

(i) *There is $\beta>0$ such that $|\widehat{d\mu}(x)|\le C\,|x|^{-\beta/2}$.*

(ii) *There is $\alpha>0$ such that $\mu(B_r(x))\le C\,r^\alpha$ for every ball $B_r(x)$ of radius $r$ centered at $x$.*

*Then, for $1\le p<\frac{2(2n-2\alpha+\beta)}{4(n-\alpha)+\beta}$, we have*

$$\left(\int|\widehat{f}|^2d\mu\right)^{\frac12}\le\ C\ \|f\|_{L^p(\mathbf{R}^n)}. \tag{5}$$

*Proof.* We have $\|\widehat{f}\|^2_{L^2(d\mu)} \leq \|f * \widehat{d\mu}\|_{p'} \; \|f\|_p$ with $p'$ the dual exponent of $p$. The theorem follows if we can show that the convolution operator $Tf = \widehat{d\mu} * f$ is bounded from $L^p \to L^{p'}$, for $p' > \frac{2(2n-2\alpha+\beta)}{\beta}$. Let $\varphi_k = \varphi(\frac{\cdot}{2^k}) \in C_0^\infty$, $0 \leq \varphi_k \leq 1$, have support in the spherical annulus $\{2^{k-1} \leq |x| \leq 2^{k+1}\}$ and define $\varphi_0 \in C_0^\infty(|x| \leq 2)$ such that $\sum_{k\geq 0} \varphi_k = 1$. We decompose $T$ according to this partition: $Tf = \sum_{k\geq 0} T_k f$, where $T_k f = (\varphi_k \widehat{d\mu}) * f$. Then by (i) we have

$$||T_k||_{L^1\to L^\infty} \leq \|\varphi_k \; \widehat{d\mu}\|_\infty \leq C \; 2^{-k\frac{\beta}{2}}. \tag{6}$$

By Plancherel's Theorem we get for the norm of the operators $T_k$ on $L^2$

$$\begin{aligned} ||T_k||_{L^2\to L^2} &\leq C \sup_{x\in\mathbf{R}^n} |\widehat{\varphi_k} * d\mu(x)| \\ &\leq C \; 2^{kn} \sup_{x\in\mathbf{R}^n} \int \frac{1}{(1+2^k|x-y|)^N} \; d\mu(y) \\ &= C' \; 2^{kn} \sup_{x\in\mathbf{R}^n} \int \int_0^\infty \chi_{B_{\frac{r}{2^k}}(x)}(y) \; (1+r)^{-N-1} \; dr \; d\mu(y) \\ &= C' \; 2^{kn} \sup_{x\in\mathbf{R}^n} \int_0^\infty \mu(B_{\frac{r}{2^k}}(x)) \; (1+r)^{-N-1} \; dr. \end{aligned}$$

Using our assumption (ii) on the regularity of the measure $\mu$, i.e. $\mu(B_{\frac{r}{2^k}}(x)) \leq C \; r^\alpha \, 2^{-k\alpha}$, we get

$$||T_k||_{L^2\to L^2} \leq C \; 2^{k(n-\alpha)}. \tag{7}$$

Interpolating the bounds (6) and (7) gives

$$||T_k||_{L^p\to L^{p'}} \leq C \; 2^{k(\frac{2n-2\alpha+\beta}{p'} - \frac{\beta}{2})}.$$

Hence, summing a geometric series we see that $T = \sum \; T_k$ is bounded from $L^p$ to $L^{p'}$ for $p' > 2\frac{2n-2\alpha+\beta}{\beta}$.

As we have seen in the previous section there exist sets $E \subset [0,1]$ supporting a measure $dF$ that satisfies regularity condition (ii) with $\alpha = \dim_H E$ and the condition on the Fourier transform in (i) with $\beta = \alpha(1 - \frac{4}{\sqrt{M}})$. Hence, we get the following result

**Corollary 0.10.** *Let $p_0 < \frac{2(2-\alpha)}{4-3\alpha}$ and choose $M$ in (4) sufficiently large. Then*

$$\int_E |\widehat{f}|^2 dF \; \leq \; C \; \|f\|^2_{L^{p_0}(\mathbf{R})},$$

*where $C$ depends only on $M$.*

**Remark 0.11.** *(1) For the uniform measure on $S^{n-1}$, Theorem 0.4.1 gives the Stein-Tomas theorem except for the endpoint $p = 2(n+1)/(n+3)$. Here $\alpha = n-1$,* *which simply follows from $S^{n-1}$ being a smooth $(n-1)$-dimensional submanifold of $\mathbf{R}^n$ and $\beta = n-1$, reflecting the fact that $S^{n-1}$ has nonvanishing curvature.*

*(2) It is an open question whether Corollary 0.4.2 holds for some $p$ in the interval $(\frac{2(2-\alpha)}{4-3\alpha}, \frac{2}{2-\alpha})$. This is interesting because in case $2/(2-\alpha)$ is the right critical exponent, a localization to scale $R$ seems to shed some new light on the construction of $\Lambda(p)$-sets.*

*(3) Suppose $E$ and $F$ are as in the corollary and, moreover, that $E$ has Hausdorff dimension $\alpha = \frac{1}{2} + \epsilon$, $\epsilon > 0$. Choosing $M$ large enough we find $\widehat{dF} \in L^4(\mathbf{R})$. Then using Plancherel's theorem we get*

$$\|\widehat{g dF}\|_4^4 = \int_{\mathbf{R}} |g\, dF * g\, dF|^2 \, dx \leq \|g\|_\infty^4 \, \|\widehat{dF}\|_4^4.$$

*A factorization argument (see [2]) shows that there is a nonnegative weight function $\omega$ with $\int_E \omega dF = 1$ such that the measure $d\mu = \omega dF$ satisfies*

$$\|\widehat{g d\mu}\|_4 \leq C \, \|g\|_{L^q(d\mu)}$$

*for $q > 4$. It would be interesting to know whether this estimate also holds for some $q < 4$.*

## 0.12 Application to multiplier theory

We proceed to construct $L^p$-multipliers on $\mathbf{R}$ which may serve as analogues of the Bochner-Riesz multipliers in $\mathbf{R}^n, n > 1$. Suppose that $E \subset [0,1]$ is a compact set of Hausdorff dimension $\alpha$ supporting a measure $\mu$ satisfying both $|\widehat{d\mu}(x)| \leq C_\beta \, |x|^{-\frac{\beta}{2}}$ and the regularity estimates $\mu(B_r(x)) \leq C \, r^\alpha$. Let $\psi \in C_0^\infty([-1,1]), \psi(0) \neq 0$, be an even function and define $k_z(x) = \frac{\psi(x)}{|x|^{\alpha - z}}, z > 0$. For $z > 0$, we will consider the multipliers given by

$$m_z(x) = \int k_z(x-y) \, d\mu(y).$$

Let $T_z$ be the convolution operator corresponding to the multiplier $m_z$. Obviously, since $k_z$ and $\mu$ have compact support, the same holds for $m_z$. Furthermore, the analysis in the proof of Theorem 0.4.1 shows that $m_z$ is a bounded measurable function provided $z > 0$. Hence, $T_z$ is bounded on $L^2(\mathbf{R})$. To get a necessary condition for $m_z$ to be a multiplier for $L^p(\mathbf{R})$, we note that $T_z \varphi = \widehat{m_z}$ for a testfunction $\varphi$ satisfying $\varphi(x) = 1$ for $x \in supp\,(m_z)$. Now, assuming $\widehat{m_z} = \widehat{d\mu}\,\widehat{k}_z \in L^p(\mathbf{R})$ and using $\widehat{k}_z(x) \approx C \, \psi(0) \, |x|^{-(1+z-\alpha)}$ and $|\widehat{d\mu}| \leq C \, |x|^{-\beta/2}$ we find that

$$\int |\widehat{d\mu}|^{p(1+\frac{2}{\beta}(1+z-\alpha))} \, dx \leq C \int |\widehat{d\mu}|^p \, |\hat{k}_z|^p \, dx < \infty.$$

Hence, Hölder's inequality gives $I_\sigma(\mu) < \infty$, for all $\sigma < \frac{2}{p(1+\frac{2}{\beta}(1+z-\alpha))}$. Since $\alpha = dim_H E \geq \sigma$ we get the necessary condition $p \geq \frac{2\beta}{\alpha(2+2z-2\alpha+\beta)}$. Note that for this condition we did not use the local regularity of the measure $\mu$.

**Theorem 0.13.** *Let $1 \leq p < \frac{2(2-2\alpha+\beta)}{4-4\alpha+\beta}$. Then $T_z$ is a bounded operator on $L^p(\mathbf{R})$ for $p > \frac{2}{2z+2-\alpha}$.*

*Proof.* We proceed as in the proof of Theorem 0.4.1, i.e. we first decompose $T_z$ into dyadic pieces in the same way as in the proof of Theorem 0.4.1; $T_z f = \sum_{k\geq 0}(\varphi_k \widehat{m_z}) * f = \sum_{k\geq 0} T_k f$. To bound the norm of each individual term $T_k$ we note, since $\varphi_k$ has support in a dyadic interval of length $2^k$, that it is enough to establish

$$\int_{\{x|\,|x|\leq 2^{k+1}\}} |T_k f|^p \, dx \leq C\; 2^{-\varepsilon k} \int_{\{x|\,|x|\leq 2^{k}\}} |f|^p \, dx$$

for some $\varepsilon > 0$. Hölders inequality implies

$$\|T_k f\|_{L^p(\{x|\,|x|<2^{k+1}\})} \quad \leq \quad C\; 2^{k(\frac{1}{p}-\frac{1}{2})}\; \|T_k f\|_{L^2(\{x|\,|x|<2^{k+1}\})}$$

and, using Plancherel's theorem, we bound the $L^2$-norm from above by

$$\begin{aligned} \|T_k f\|^2_{L^2(\mathbf{R})} &= \|\varphi_k \widehat{k_z}\widehat{d\mu} * f\|_2^2 \\ &= \|\{(\varphi_k \widehat{k_z})^\vee *\; d\mu\}\; \widehat{f}\,\|_2^2 \\ &\leq \|\; |(\varphi_k \widehat{k_z})^\vee| * d\mu\,\|_\infty \quad \|\; \{|(\varphi_k \widehat{k_z})^\vee| *\; d\mu\}\; |\widehat{f}\,|^2\,\|_1. \end{aligned}$$

Now, by Fubini's theorem the above $L^1$-norm equals $\int_{\mathbf{R}}\; |(\varphi_k \widehat{k_z})^\vee(x)\;|\;\{d\mu * |\widehat{f}\,|^2(x)\,\}\; dx$, which we estimate further by

$$\|(\varphi_k \widehat{k_z})^\vee\|_1 \quad \sup_x \int\; |\widehat{f}(x-y)|^2 d\mu(y).$$

Using translation invariance we apply Theorem0.4.1 and see that the last integral is bounded by $\|f\|_p^2$. Hence

$$\|T_k f\|_2^2 \leq C\; \|\;|(\varphi_k \widehat{k_z})^\vee| * d\mu\,\|_\infty \quad \|(\varphi_k \widehat{k_z})^\vee\|_1 \quad \|f\|_p^2.$$

Now, since $\widehat{k_z}$ is a smooth function which decays like $|x|^{-(1+z-\alpha)}$, we may assume that $\Phi_k = 2^{k(1+z-\alpha)}\varphi_k \widehat{k_z}$ is a function which shares the same properties as $\varphi_k$. As in the proof of Theorem 0.4.1 we obtain

$$\|\;|(\varphi_k \widehat{k_z})^\vee| * d\mu\,\|_\infty \leq 2^{-k(1+z-\alpha)}\; \|\;|\Phi_k^\vee| * d\mu\|_\infty \leq C\; 2^{-k(1+z-\alpha)}\; 2^{k(1-\alpha)}$$

and $\|(\varphi_k \widehat{k_z})^\vee\|_1 \leq 2^{-k(1+z-\alpha)}\; \|\Phi_k^\vee\|_1 \leq C\; 2^{-k(1+z-\alpha)}$. Collecting terms gives

$$\|T_k f\|_{L^p} \leq 2^{k(-1-z+\alpha/2+\frac{1}{p})}\|f\|_p \tag{8}$$

and, by summing a geometric series we see, for $p > \frac{2}{2z+2-\alpha}$, that the operator $T_z$ is bounded on $L^p(\mathbf{R})$.

Obviously there is a gap between our necessary and sufficient conditions for $m_z$ to be a multiplier for $L^p(\mathbf{R})$. However, since this gap can be made arbitrarily small we get:

**Corollary 0.14.** *Let $I$ be an interval and let $M_p(I)$ be the algebra consisting of all $L^p(\mathbf{R})$-multipliers with support in $I$. Then $M_p(I) \neq M_q(I)$ for $1 \leq p < q < 2$.*

**Remark 0.15.** *In the same way as above we may construct multipliers $m_z = k_z * d\mu, z > 0$, with $d\mu$ satisfying a better estimate on the Fourier transform side as noted in Remark 0.3.4. Then $p \geq \frac{2}{2z+2-\alpha}$ is a necessary condition for $m_z$ to be a multiplier for $L^p(\mathbf{R})$ and, because the weaker local regularity estimate $\mu(B_r(x)) \leq Cr^\alpha(1 + |\log r|)$ causes only a power in $k$ in the bound (8), one may close the mentioned gap in this case. However, the case of the critical line $p = \frac{2}{2z+2-\alpha}$ remains unanswered.*

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