Optimal extension of the Hausdorff-Young inequality

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Abstract

Given $1 < p < 2$, we construct a Banach function space $F^p(T)$ with $\sigma$-order continuous norm which contains $L^p(T)$ and has the property that the Fourier transform map $F : L^p(T) \to \ell^{p'}(Z)$ has a continuous $\ell^{p'}(Z)$-valued extension to $F^p(T)$. Moreover, $F^p(T)$ is maximal with these properties and satisfies $L^p(T) \subset F^p(T) \subset L^1(T)$ with both containments proper. Each $F^p(T)$ turns out to be a weakly sequentially complete, translation invariant, homogeneous Banach space and consists precisely of those functions $f \in L^1(T)$ such that $\hat{f}_{\chi_A} \in \ell^{p'}(Z)$ for every Borel set $A \subset T$. This answers a question of R.E. Edwards posed some 40 years ago.

1 Introduction and main results

It is known that for $1 \leq p \leq 2$ the Fourier transform $F$ maps $L^p(T)$ into $\ell^{p'}(Z)$, where $1/p' + 1/p = 1$, and the Hausdorff-Young inequality

$$\|\hat{f}\|_{p'} \leq \|f\|_p, \quad f \in L^p(T)$$

ensures that $F$ is continuous. Moreover, the Fourier transform is an injective map from distributions $D(T)$ into the space of sequences of polynomial growth. The theme of this note is to address the following question: Is the Hausdorff-Young inequality optimal? That is, keeping the range space $\ell^{p'}(Z)$ fixed, is it possible to continuously extend the Fourier transform operator $F : L^p(T) \to \ell^{p'}(Z)$ to a Banach function space $F^p(T)$, over the probability space $(T, \mathcal{B}(T), dt)$ (see Section 2 for the definition), which is larger than $L^p(T)$ and in such a way that $F^p(T)$ is maximal (or optimal) with this property? Moreover, if so, can $F^p(T)$ be identified in some "concrete" way? That there exist distributions which are not in $L^p(T)$, but whose Fourier transform lies in $\ell^{p'}(Z)$, is known. Here are some examples (see [St, p.339],[Zyg, II, p.102]). For almost all sign changes $\sum_{n \geq 1} \pm n^{-1/2} \cos nx$ is not integrable while its Fourier transform is in all $\ell^{p'}(Z)$ for $p' > 2$. An even more concrete example is a function with a Riemann singularity of order $0 < \lambda < 2$ at $0$, say

$$f_\lambda(x) = e^{i/x} x^{-\lambda}, \quad 0 < x < 1$$

and $f_\lambda(x) = 0$ elsewhere in $(-\pi, \pi]$, which has Fourier transform $\hat{f}_\lambda(n) = \sqrt{\pi} e^{3\sqrt{n}} n^{-3/4+\lambda/2} + O(n^{-1+\lambda/2})$ if $n \to +\infty$ and decays like the power given in the $O$-term for $n \to -\infty$. Hence,
\[ \hat{f}_\lambda \text{ lies in } \ell^p(Z) \text{ for certain } p' > 2 \text{ depending on } \lambda \text{ while, for } \lambda > 1, \text{ the function } f_\lambda \in L^0(T) \text{ is not integrable at } 0. \] Here, \( L^0(T) \) denotes the space of all complex-valued \( \mathcal{B}(T) \)-measurable (i.e. Borel measurable) functions on \( T \).

**Theorem 1.1.** Let \( 1 \leq p \leq 2 \). There exists a Banach function space \( F^p(T) \subset L^0(T) \) with \( \sigma \)-order continuous norm \( \| \cdot \|_{F^p(T)} \) and having the following properties:

(i) \( L^p(T) \) is continuously included in \( F^p(T) \) and the Fourier transform map \( F : L^p(T) \to \ell^p(Z) \) has an extension to a continuous linear operator from \( F^p(T) \) into \( \ell^p(Z) \). More precisely,

\[
\|f\|_{F^p(T)} \leq 4 \|f\|_p, \quad f \in L^p(T).
\]

(ii) If \( Z \) is any Banach function space over \((T, \mathcal{B}(T), dt)\) with \( \sigma \)-order continuous norm such that \( L^p(T) \) is continuously included in \( Z \) and \( F \) has an extension to a continuous linear operator from \( Z \) into \( \ell^p(Z) \), then \( Z \) is continuously included in \( F^p(T) \).

(iii) \( F^p(T) \subset L^1(T) \) with \( \|f\|_1 \leq \|f\|_{F^p(T)} \). Moreover, the \( \ell^p(Z) \)-valued extension of \( F \) from \( L^p(T) \) to \( F^p(T) \) is again the map \( f \to \hat{f} \) for \( f \in F^p(T) \).

Theorem 1.1 justifies the statement (within a well defined and extensive class of spaces) that the Hausdorff-Young inequality has an \( \ell^p(Z) \)-valued extension to a larger maximal domain \( F^p(T) \), which we will call its optimal lattice domain. By (ii), \( F^p(T) \) is unique up to isomorphism; its norm turns out to be

\[
\|f\|_{F^p(T)} = \sup \left\{ \int_T |f| |\hat{\phi}| \; dt : \phi \in \ell^p(Z), \|\phi\|_p = 1 \right\}.
\]

According to (iii), neither the random series mentioned above nor the functions (1.1) with Riemann singularities for \( 1 < \lambda < 2 \) are contained in \( F^p(T) \). We remark that the above mentioned result and those below are also valid for higher dimensional tori \( \mathbb{T}^d, d > 1 \). It will become apparent in the sequel that the restriction conjecture for the Fourier transform (see [St2]) can be rephrased as finding sharp bounds for the \( F^p(\mathbb{T}^d) \)-norm of smooth bump functions with support in a \( \delta \)-neighbourhood of the \( d \)-dimensional unit sphere.

We now turn to more concrete descriptions of \( F^p(T) \). Given \( 1 \leq p \leq 2 \), define a vector subspace \( V^p(T) \) of \( \ell^p(Z) \) by

\[
V^p(T) = \{ h \in \ell^p(T) : h = \hat{\phi} \text{ for some } \phi \in \ell^p(Z) \}. \tag{1.2}
\]

For each \( f \in L^1(T) \) define a linear map \( S_f : L^\infty(T) \to c_0(Z) \) by

\[
S_f : g \mapsto \hat{f}g, \quad g \in L^\infty(T). \tag{1.3}
\]

Clearly \( S_f \) is continuous with operator norm \( \|S_f\| \leq \|f\|_1 \). For each \( 1 \leq p \leq 2 \), let \( L(L^\infty(T), \ell^p(Z)) \) denote the Banach space of all continuous operators \( T : L^\infty(T) \to \ell^p(Z) \) equipped with the operator norm

\[
\|T\|_{\infty,p'} = \sup_{\|g\|_\infty = 1} \|Tg\|_{p'}. \]
If \( f \in L^1(\mathbb{T}) \) has the property that \( \text{Range}(S_f) \subset \ell^r(\mathbb{Z}) \), then the closed graph theorem implies that \( \|S_f\|_{\infty,\ell^r} < \infty \).

**Theorem 1.2.** Let \( 1 \leq p \leq 2 \). Each of the spaces

\[
\Delta^p(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \int_{\mathbb{T}} |f| \, dt < \infty, \forall g \in V^p(\mathbb{T})\},
\]

\[
\Phi^p(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{\int_X A} \in \ell^r(\mathbb{Z}), \forall A \in B(\mathbb{T})\},
\]

\[
\Gamma^p(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \text{Range}(S_f) \subset \ell^r(\mathbb{Z})\},
\]

coincides with the optimal lattice domain \( F^p(\mathbb{T}) \) of the Hausdorff-Young inequality. Moreover, in the case of (1.6), the operator norm \( \|S_f\|_{\infty,\ell^r} \) is equivalent to the norm of \( f \) in \( F^p(\mathbb{T}) \), for each \( f \in F^p(\mathbb{T}) \).

**Remark 1.3.**

(i) For \( p = 1 \) it turns out that \( F^1(\mathbb{T}) = L^1(\mathbb{T}) \) and for \( p = 2 \) that \( F^2(\mathbb{T}) = L^2(\mathbb{T}) \). So, both the Fourier transform maps \( F : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z}) \) and \( F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}) \) are already defined on their optimal domain. Also, for \( 1 \leq p < q \leq 2 \) we have \( \ell^r(\mathbb{Z}) \subset \ell^q(\mathbb{Z}) \) and therefore \( V^p(\mathbb{T}) \subset V^q(\mathbb{T}) \subset L^2(\mathbb{T}) \). It is then clear from (1.4) that \( L^2(\mathbb{T}) \subset F^q(\mathbb{T}) \subset F^p(\mathbb{T}) \subset L^1(\mathbb{T}) \).

(ii) It is not obvious from (1.5) that the space \( \Phi^p(\mathbb{T}) \) is actually an ideal relative to the pointwise a.e. order in \( L^0(\mathbb{T}) \). That is, if \( f \in \Phi^p(\mathbb{T}) \) and \( g \in L^0(\mathbb{T}) \) satisfies \( |g| \leq |f| \) a.e., then also \( g \in \Phi^p(\mathbb{T}) \). Of course, being equal to the Banach function space \( F^p(\mathbb{T}) \), it must have this property. In addition to having \( \sigma \)-order continuous norm, it will be seen that the optimal domain \( F^p(\mathbb{T}) \) has other desirable properties; it is translation invariant, weakly sequentially complete, has the \( \sigma \)-Fatou property, etc. (see the end of Section 3).

(iii) For \( 1 \leq p \leq 2 \), the following question was raised some forty years ago by R.E. Edwards, [Ed, p.206]: What can be said about the family of functions \( f \in L^1(\mathbb{T}) \) having the property that \( \widehat{f \chi_A} \) lies in \( \ell^r(\mathbb{Z}) \) for all \( A \in B(\mathbb{T}) \)? Theorems 1.1 and 1.2 provide an exact answer: this family of functions is precisely the optimal lattice domain \( F^p(\mathbb{T}) \) of the Fourier transform map \( F : L^p(\mathbb{T}) \to \ell^r(\mathbb{Z}) \).

**Remark 1.3(iii)** raises the question of whether \( F^p(\mathbb{T}) \) is genuinely larger than \( L^p(\mathbb{T}) \).

**Theorem 1.4.** Let \( 1 < p < 2 \). Then both the inclusions \( L^p(\mathbb{T}) \subset F^p(\mathbb{T}) \subset L^1(\mathbb{T}) \) are proper.

**Remark 1.5.** It is known that there exists \( f \in L^1(\mathbb{T}) \) whose Fourier transform does not lie in \( \ell^r(\mathbb{Z}) \) for any \( p < \infty \), e.g. \( f(t) = \sum_{n=2}^{\infty} \frac{\cos nt}{\log n} \) has this property. Accordingly, the inclusion \( F^p(\mathbb{T}) \subset L^1(\mathbb{T}) \) is always proper. That the other inclusion is also proper will be established in Section 4.

What is the connection between the (apparently) abstract space \( F^p(\mathbb{T}) \) in the statement of Theorem 1.1 with the more concrete descriptions of \( F^p(\mathbb{T}) \) given in Theorem 1.2? It is routine to check that the set function \( m_p : B(\mathbb{T}) \to \ell^r(\mathbb{Z}) \) defined by

\[
m_p : A \mapsto F(\chi_A), \quad A \in B(\mathbb{T}),
\]

(1.7)
is $\sigma$-additive, that is, it is an $\ell^p'(\mathbb{Z})$-valued vector measure. Moreover, the $m_p$-null sets coincide with the Lebesgue null sets in $\mathbb{T}$. This crucial point allows us to view the Banach lattice $L^1(m_p)$ of all $m_p$-integrable functions (modulo $m_p$-null functions) as a Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$. It is precisely the space $L^1(m_p)$ which is proved to have the optimality properties required of $F_p(\mathbb{T})$ in Theorem 1.1. That is, $F_p(\mathbb{T}) = L^1(m_p)$ and the integration map $f \mapsto \int_{\mathbb{T}} f dm_p$, from $L^p(\mathbb{T})$ to $\ell^p'(\mathbb{Z})$, is precisely the continuous extension of $F$ from $L^p(\mathbb{T})$ to $F_p(\mathbb{T})$.

This approach to optimal extensions, via the integration map of appropriate vector measures, has proved to be effective in recent years in the treatment of various operators/inequalities arising in classical analysis; see for example [CR1],[CR2],[CR3],[CR4],[OR1],[OR2] and the references therein. For a different extension of the Fourier transform we refer to [Gu] and the references therein.

2 Proof of Theorem 1.1

We begin with some preliminaries concerning integration with respect to a general vector measure. A set function $m : \Sigma \to X$, where $X$ is a complex Banach space and $\Sigma$ is a $\sigma$-algebra of subsets of a non-empty set $\Omega$, is called a vector measure if it is $\sigma$-additive, that is, $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ for all pairwise disjoint sequences $\{A_n\}_{n=1}^{\infty}$ in $\Sigma$ with the series being (norm) unconditionally convergent in $X$. A set $A \in \Sigma$ is called $m$-null if $m(B) = 0$ for all $B \in \Sigma$ which are contained in $A$. The variation $|m|$ of $m$ is the smallest, $[0, \infty]$-valued measure satisfying $\|m(A)\| \leq |m|(A)$, for all $A \in \Sigma$, and can be defined (as for scalar measures) via finite partitions, [DU, Ch.I]. If $|m|(\Omega) < \infty$, then $m$ is said to have finite variation. The semi-variation of $m$ is the set function $\|m\| : \Sigma \to [0, \infty)$ defined by

$$\|m\|(A) := \sup_{x' \in X', \|x'\|=1} |\langle m, x' \rangle|(A), \quad A \in \Sigma,$$

(2.1)

where $X'$ is the dual Banach space of $X$ and $\langle m, x \rangle$ denotes the complex measure $A \mapsto \langle m(A), x' \rangle$ on $\Sigma$, for each $x' \in X'$. Then

$$\sup_{B \in \Sigma, B \subset A} \|m(B)\| \leq \|m\|(A) \leq 4 \sup_{B \in \Sigma, B \subset A} \|m(B)\|$$

(2.2)

for each $A \in \Sigma$, [DU, p.4]. The vector measure $m$ is said to have relatively compact range if the closure of its range $m(\Sigma) := \{m(A) : A \in \Sigma\}$ is a compact subset of $X$.

A $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is called $m$-integrable if

$$\int_{\Omega} |f| \, d|\langle m, x' \rangle| < \infty, \quad x' \in X',$$

(2.3)

and for each $A \in \Sigma$ there exists a vector in $X$, necessarily unique and denoted by $\int_A f \, dm$, such that

$$\langle \int_A f \, dm, x' \rangle = \int_A f \, d\langle m, x' \rangle, \quad x' \in X'.$$

(2.4)
By the Orlicz-Pettis theorem, [DU, p.22], the set function
\[ m_f : A \mapsto \int_A f \, dm, \quad A \in \Sigma, \]  
(2.5)
is also an \( X \)-valued vector measure. The linear space of all \( m \)-integrable functions is denoted by \( L^1(m) \); it is equipped with the lattice seminorm
\[ \| f \|_{L^1(m)} := \sup_{x' \in X', \| x' \|=1} \int_{\Omega} |f| (m, x'). \]  
(2.6)

Note that \( \| f \|_{L^1(m)} = \| m_f \| (\Omega) \), where \( \| m_f \| (\cdot) \) is the semi-variation of the vector measure \( m_f \).

It follows from (2.2) applied to \( m_f \) that
\[ \sup_{A \in \Sigma} \left\| \int_A f \, dm \right\| \leq \| f \|_{L^1(m)} \leq 4 \sup_{A \in \Sigma} \int_A f \, dm \]  
(2.7)
for every \( f \in L^1(m) \). A function \( f \in L^1(m) \) is called \( m \)-null if \( \| f \|_{L^1(m)} = 0 \) or equivalently, if \( m_f \) is the zero vector measure. The quotient space of \( L^1(m) \) modulo \( m \)-null functions is again identified with (and denoted by) \( L^1(m) \). Then \( L^1(m) \) is a complex Banach lattice relative to the pointwise \( m \)-a.e. order on \( \Omega \) and the norm given by (2.6). That is, \( |f| \leq |g| \) \( m \)-a.e. implies \( \| f \|_{L^1(m)} \leq \| g \|_{L^1(m)} \). Moreover, the norm is \( \sigma \)-order continuous (as a consequence) of the dominated convergence theorem, [Le2, Theorem 2.2], meaning that if non-negative functions \( f_n \) decrease to 0 as \( n \to \infty \) in the order of \( L^1(m) \), then \( f_n \to 0 \) in \( L^1(m) \) as \( n \to \infty \). For these claims we refer to [FNR] and the references therein. Moreover, every \( \Sigma \)-measurable function \( f : \Omega \to \mathbb{C} \) which satisfies \( |f| \leq K, \) \( m \)-a.e., for some \( K > 0 \) (that is \( f \in L^\infty(m) \)), is necessarily \( m \)-integrable, [Le2, Theorem 2.2], and satisfies (via (2.6))
\[ \| f \|_{L^1(m)} \leq \| f \|_{L^\infty(m)} \| m \| (\Omega). \]  
(2.8)

It follows from (2.7) that the integration map \( I_m : f \mapsto \int_{\Omega} f \, dm \) is a continuous linear operator from \( L^1(m) \) into \( X \) with operator norm \( \| I_m \| = 1 \).

It is time to specialize to the particular vector measure (1.7).

**Lemma 2.1.** Let \( 1 \leq p \leq 2 \).

(i) The vector measure \( m_p : \mathcal{B}(\mathbb{T}) \to \ell^p (\mathbb{Z}) \) as given by (1.7) is mutually absolutely continuous with respect to Lebesgue measure on \( \mathbb{T} \) (i.e. \( m_p \)-null sets are Lebesgue null sets).

(ii) For \( 1 < p \leq 2 \), \( m_p \) has infinite variation.

(iii) The containment \( L^1(m_p) \subset L^1(\mathbb{T}) \) is valid with
\[ \| f \|_1 \leq \| f \|_{L^1(m_p)}, \quad f \in L^1(m_p). \]  
(2.9)

Moreover, \( L^1(m_p) \) is dense in \( L^1(\mathbb{T}) \).
Proof. (i) is clear from (1.7) and the definition of the Fourier transform. For, if \( A \in \mathcal{B}(\mathbb{T}) \) has Lebesgue measure zero, then \( \hat{\chi}_B = 0 \) for all \( B \in \Sigma \) with \( B \subset A \). So, \( A \) is \( m_p \)-null. On the other hand, suppose that \( A \in \mathcal{B}(\mathbb{T}) \) is \( m_p \)-null. Then, in particular, \( \hat{\chi}_A = 0 \) and hence, by injectivity of the Fourier transform, \( \chi_A = 0 \) in \( L^p(\mathbb{T}) \), that is, \( |A| = 0 \).

(ii) Fix 1 < \( p \leq 2 \) and an integer \( N > 1 \). Set \( A_j = [2\pi(j-1)/N, 2\pi j/N) \) for 1 \( \leq j \leq N \). Then the sets \( A_j \) are pairwise disjoint and \( |A_j| = 1/N \) for each 1 \( \leq j \leq N \). It is routine to check that

\[
|\hat{\chi}_{A_j}(m)| = 1/N, \quad 0 \leq m < N,
\]

and hence, that \( \|\hat{\chi}_{A_j}\|_{\ell^p(\mathbb{Z})} \geq N^{-1/p} \). Since each function \( \chi_{A_j} \) is a translate of \( \chi_{A_1} \) for 1 < \( j \leq N \), it follows that

\[
\sum_{j=1}^{N} \|m_p(A_j)\|_{p'} = \sum_{j=1}^{N} \|\hat{\chi}_{A_j}\|_{p'} \geq N^{1-1/p}.
\]

Hence, \( m_p \) must have infinite variation.

(iii) We have, for \( \chi_{\{0\}} \in (\ell^p(\mathbb{Z}))' = \ell^q(\mathbb{Z}) \), that

\[
\langle m_p(A), \chi_{\{0\}} \rangle = \hat{\chi}_A(0) = |A|
\]

that is

\[
|A| = |\langle m_p, \chi_{\{0\}} \rangle| (A), \quad A \in \mathcal{B}(\mathbb{T}).
\]

According to (2.6), if \( f \in L^1(m_p) \), then \( \int_{\mathbb{T}} |f| \, dt = \int_{\mathbb{T}} |f| \, d\langle m_p, \chi_{\{0\}} \rangle \) < \( \infty \) and so \( f \in L^1(\mathbb{T}) \). Moreover, since \( \|\chi_{\{0\}}\|_{\ell^p(\mathbb{Z})} = 1 \), we obtain from (2.6) that

\[
\|f\|_1 = \int_{\mathbb{T}} |f| \, dt = \int_{\mathbb{T}} |f| \, d\langle m_p, \chi_{\{0\}} \rangle \leq \|f\|_{L^1(m_p)}.
\]

This establishes \( L^1(m_p) \subset L^1(\mathbb{T}) \).

According to (i), the \( \mathcal{B}(\mathbb{T}) \)-simple functions in \( L^1(m_p) \) coincide with those in \( L^1(\mathbb{T}) \). Hence, \( L^1(m_p) \) is dense in \( L^1(\mathbb{T}) \). \( \Box \)

Remark 2.2. For \( p = 1 \) the vector measure \( m_1 \) does have finite variation. Indeed, let \( \{A_k\} \) be a Borel partition of \( \mathbb{T} \). Then

\[
\sum_k \|m_1(A_k)\|_{\infty} = \sum_k \|\hat{\chi}_{A_k}\|_{\infty} \leq \sum_k |A_k| = 1
\]

and so \( |m_1|(\mathbb{T}) \) is finite.

A sublattice \( Z \) of \( L^0(\mathbb{T}) \) is an ideal if every \( f \in L^0(\mathbb{T}) \) satisfying \( |f| \leq |h| \) for some \( h \in Z \) is necessarily itself in \( Z \). If, in addition, there is a norm on \( Z \) such that \( Z \) is a Banach lattice relative to this norm, for the order induced from \( L^0(\mathbb{T}) \), then \( Z \) is called a Banach function space over \( (\mathbb{T}, \mathcal{B}(\mathbb{T}), dt) \); see [Za, Ch.15]. Since the \( m_p \)-null sets and the Lebesgue null sets coincide, the previous recorded properties of the spaces \( L^1(m) \), with \( m \) a general vector measure, when specialized to \( L^1(m_p) \) imply that \( L^1(m_p) \) is a Banach function space over \( (\mathbb{T}, \mathcal{B}(\mathbb{T}), dt) \).
Lemma 2.3. Let $1 \leq p \leq 2$. Then $L^p(\mathbb{T}) \subset L^1(m_p)$ and
\[
\int_A f \, dm_p = \hat{\chi}_A, \quad A \in \mathcal{B}(\mathbb{T}),
\] 
for every $f \in L^p(\mathbb{T})$. Moreover, we have
\[
\|f\|_{L^1(m_p)} \leq 4 \|f\|_p, \quad f \in L^p(\mathbb{T}).
\] 
Proof. To verify $L^p(\mathbb{T}) \subset L^1(m_p)$ it suffices to show that non-negative functions $f \in L^p(\mathbb{T})$ belong to $L^1(m_p)$. Choose simple functions $\{s_n\}_{n \in \mathbb{N}}$ with $0 \leq s_n \uparrow f$ pointwise on $\mathbb{T}$ and fix $A \in \mathcal{B}(\mathbb{T})$. Since $L^p(\mathbb{T})$ has $\sigma$-order continuous norm, we conclude that $\chi_A s_n \to \chi_A f$ in $L^p(\mathbb{T})$ as $n \to \infty$. By continuity of the Fourier transform map we obtain $\hat{\chi}_A s_n \to \hat{\chi}_A f$ in $\ell^p(\mathbb{Z})$ as $n \to \infty$. It is routine to check that
\[
\int_B h \, dm_p = \hat{\chi}_B \hat{h}, \quad B \in \mathcal{B}(\mathbb{T}),
\] 
for every $\mathcal{B}(\mathbb{T})$-simple function $h$ on $\mathbb{T}$. Hence, $\int_A s_n \, dm_p \to \hat{\chi}_A f$ in $\ell^p(\mathbb{Z})$ as $n \to \infty$. According to [Le2, Theorem 2.4] the function $f \in L^1(m_p)$ and (2.10) holds.

To establish (2.11), let $f \in L^p(\mathbb{T})$. According to (2.7) and (2.10) we have
\[
\|f\|_{L^1(m_p)} \leq 4 \sup_{A \in \mathcal{B}(\mathbb{T})} \|\hat{\chi}_A f\|_{\ell^p}.
\] 
By the Hausdorff-Young inequality
\[
\|\hat{\chi}_A f\|_{\ell^p} \leq \|\chi_A f\|_p \leq \|f\|_p, \quad A \in \mathcal{B}(\mathbb{T}).
\] 
Hence, (2.11) holds. \qed

Corollary 2.4. Let $1 \leq p \leq 2$. Then, for every $f \in L^1(m_p)$ and $A \in \mathcal{B}(\mathbb{T})$, we have
\[
\int_A f \, dm_p = \hat{\chi}_A f.
\] 
In particular, the integration map $I_{m_p}$ is a continuous extension of $F$ from $L^p(\mathbb{T})$ to $L^1(m_p)$, still with values in $\ell^p(\mathbb{Z})$.

Proof. Fix $f \in L^1(m_p)$. Choose simple functions $\{s_n\}_{n \in \mathbb{N}}$ with $s_n \to f$ in $L^1(m_p)$ as $n \to \infty$. By continuity of the integration map $I_{m_p} : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ and (2.10) we have, for $A \in \mathcal{B}(\mathbb{T})$,
\[
\lim_{n \to \infty} \hat{\chi}_{A s_n} = \lim_{n \to \infty} \int_A s_n \, dm_p = \int_A f \, dm_p,
\] 
with convergence in $\ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$. On the other hand, (2.9) implies that $\chi_{A s_n} \to \chi_A f$ in $L^1(\mathbb{T})$ as $n \to \infty$ and so the continuity of $F : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ yields, for $A \in \mathcal{B}(\mathbb{T})$, that
\[
\lim_{n \to \infty} \hat{\chi}_{A s_n} = \hat{\chi}_A f,
\] 
with convergence in $\ell^\infty(\mathbb{Z})$. By uniqueness of Fourier transforms we see that (2.12) holds. \qed
Corollary 2.5. Let $1 \leq p \leq 2$. Then the vector measure $m_p : B(\mathbb{T}) \to \ell^p(\mathbb{Z})$ does not have relatively compact range.

Proof. The closed convex hull of $m_p(B(\mathbb{T}))$ is given by

$$C := \overline{co} m_p(B(\mathbb{T})) = \{ \int_{\mathbb{T}} f \, dm_p : 0 \leq f \leq 1, \ f \in L^\infty(m_p) \},$$

[DU, p.263]. Moreover, according to (2.12) each character $e_n$, for $n \in \mathbb{Z}$, satisfies

$$\chi_{\{n\}} = F(e_n) = \int_{\mathbb{T}} e_n \, dm_p \in C + C + iC + iC.$$  

So, if $m_p(B)$ is relatively compact in $\ell^p(\mathbb{Z})$, then so is $C + C + iC + iC$ and hence, also $\{\chi_{\{n\}} : n \in \mathbb{Z}\}$. But, this is surely not the case as $\|\chi_{\{n\}} - \chi_{\{k\}}\|_{\ell^p} = 2^{1/p'}$ for $n \neq k$. \hfill \Box

Proof of Theorem 1.1 We show that $F_p(\mathbb{T}) := L^1(m_p)$, equipped with the norm $\| \cdot \|_{F_p(\mathbb{T})} := \| \cdot \|_{L^1(m_p)}$, has all the required features. As already noted, $L^1(m_p)$ is a Banach function space over $(\mathbb{T}, B(\mathbb{T}), dt)$ with $\sigma$-order continuous norm. Part (i) of Theorem 1.1 is immediate from Lemma 2.3 and Corollary 2.4 and part (iii) is clear from Lemma 2.1 (iii).

To establish (ii), let $Z$ be any Banach function space over $(\mathbb{T}, B(\mathbb{T}), dt)$ with $\sigma$-order continuous norm such that $L^p(\mathbb{T}) \subset Z$ continuously and $F$ has a continuous linear extension $T : Z \to \ell^{p'}(\mathbb{Z})$. Let $0 \leq f \in Z$. Choose simple functions $\{s_n\}_{n \in \mathbb{N}}$ such that $0 \leq s_n \uparrow f$ pointwise a.e. on $\mathbb{T}$ and note that $\{s_n\}_{n \in \mathbb{N}} \subset L^p(\mathbb{T}) \subset Z$. Fix $A \in B(\mathbb{T})$. Since $Z$ has $\sigma$-order continuous norm, it follows that $T(s_n\chi_A) \to T(f\chi_A)$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. But, for $n \in \mathbb{N}$ we have

$$T(s_n\chi_A) = F(s_n\chi_A) = \int_A s_n \, dm_p,$$

(2.13) and so $\int_A s_n \, dm_p \to T(f\chi_A)$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. Again by [Le2, Theorem 2.4] it follows that $f \in L^1(m_p)$ and

$$\int_A f \, dm_p = \lim_{n \to \infty} \int_A s_n \, dm_p = T(f\chi_A).$$

(2.14) The case for general $f \in Z$ follows by considering the positive and negative parts of both $Re(f)$ and $Im(f)$, all of which belong to $Z$. So, $Z \subset L^1(m_p)$. It remains to verify the continuity of this inclusion. Given $f \in Z$, it follows from (2.7) and (2.14) that

$$\|f\|_{L^1(m_p)} \leq 4 \sup_{A \in B(\mathbb{T})} \|f\|_{\ell^p} = 4 \sup_{A \in B(\mathbb{T})} \|T(\chi_A f)\|_{\ell^{p'}}.$$  

By continuity, $\|T(\chi_A f)\|_{\ell^{p'}} \leq \|T\| \|f\chi_A\|_Z$ and, since the norm on $Z$ is a lattice norm and $|f\chi_A| \leq |f|$, also $\|f\chi_A\|_Z \leq \|f\|_Z$ for each $A \in B(\mathbb{T})$. It follows that

$$\|f\|_{L^1(m_p)} \leq 4 \|T\| \|f\|_Z.$$  

This completes the proof of Theorem 1.1.
Remark 2.6. We can now justify the claims made in Remark 1.3. Since $L^1(m_1) = F^1(\mathbb{T})$ and also $L^1(m_1) = L^1(\mathbb{T})$, by Lemma 2.1 (iii) and Lemma 2.3 (with equivalent norms), it follows that $F^1(\mathbb{T}) = L^1(\mathbb{T})$.

For $p = 2$, the Plancherel theorem and (2.11) yield, for each $f \in L^2(\mathbb{T})$, that
\[
\|f\|_{L^1(m_2)} \leq 4 \|f\|_2 = 4 \|\hat{f}\|_2 \leq 4 \sup_{A \in \mathcal{B}(\mathbb{T})} \|\chi_A f\|_2.
\]
Then apply (2.7) and (2.12) to conclude, for $f \in L^2(\mathbb{T})$, that
\[
\|f\|_{L^1(m_2)} \leq 4 \|f\|_2 \leq 4 \|f\|_{L^1(m_2)}.
\]
Moreover, by Lemma 2.3, $L^2(\mathbb{T})$ is contained and dense in $L^1(m_2)$. It follows that $L^1(m_2) = L^2(\mathbb{T})$.

3 Proof of Theorem 1.2

To describe the space $L^1(m)$, for a general vector measure $m$, is rather difficult. However, for the vector measures $m_p$, with $1 \leq p \leq 2$, it will be shown in this section that this is possible.

According to (1.2), $V^p(\mathbb{T}) = \{h \in L^p(\mathbb{T}) : h = \hat{\phi}$ for some $\phi \in \ell^p(\mathbb{Z})\}$. Since $1 \leq p \leq 2$, we have $\ell^p(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and so the inverse Fourier transform $\hat{\phi} \in L^2(\mathbb{T})$ for $\phi \in \ell^p(\mathbb{Z})$. For $1 \leq p < 2$ the containment $V^p(\mathbb{T}) \subset L^p(\mathbb{T})$ (3.1) is always proper; see [Ka, p.101] for $1 < p < 2$. For $p = 1$, note that $V^1(\mathbb{T}) = \{\hat{\phi} : \phi \in \ell^1(\mathbb{Z})\} \subset C(\mathbb{T})$ with a proper containment, [Ka, p.31].

Lemma 3.1. Let $1 \leq p \leq 2$ and $f \in L^0(\mathbb{T})$. Then $f \in L^1(m_p)$ if and only if
\[
\int_\mathbb{T} |f| |h| \, dt < \infty, \quad h \in V^p(\mathbb{T}).
\]

Proof. Suppose $f \in L^1(m_p)$. If $h = \hat{\phi} \in V^p(\mathbb{T})$ for some $\phi \in \ell^p(\mathbb{Z})$, then (2.3) implies that $\int_\mathbb{T} |f| \, d|\langle m_p, \phi \rangle| < \infty$. Since $L^p(\mathbb{T}) \subset L^2(\mathbb{T})$, we can apply Parseval’s formula to conclude, for each $A \in \mathcal{B}(\mathbb{T})$, that
\[
\langle m_p(A), \phi \rangle = \langle \hat{\chi_A}, \hat{h} \rangle = \langle \chi_A, \tilde{h} \rangle = \int_A \tilde{h} \, dt,
\]
where $\tilde{h}(t) = h(-t)$ is the reflection of $h$. Accordingly, the variation measure $|\langle m_p, \phi \rangle|(A) = \int_A |\hat{h}| \, dt$ for $A \in \mathcal{B}(\mathbb{T})$ and therefore
\[
\int_\mathbb{T} |f| |\tilde{h}| \, dt = \int_\mathbb{T} |f| \, d|\langle m_p, \phi \rangle| < \infty.
\]
Since \( V^p(\mathbb{T}) \) is invariant under formation of reflections, (3.2) holds. Conversely, let \( 1 < p \leq 2 \) and suppose that \( f \in L^1(\mathbb{T}) \) satisfies (3.2). Given any \( \phi \in \ell^p(\mathbb{Z}) \) there exists \( h \in L^p(\mathbb{T}) \) such that \( \hat{h} = \phi \), [Ka, IV Theorem 2.2]. Then \( h \in V^p(\mathbb{T}) \) and hence, also \( \hat{h} \in V^p(\mathbb{T}) \). So, \( \int_T |f| |\hat{h}| \ dt < \infty \). Moreover, the same calculation as above shows that the equality in (3.3) holds and hence, is finite. So, \( f \) satisfies (2.3). Since the reflexive space \( \ell^p(\mathbb{Z}) \) cannot contain an isomorphic copy of the Banach space \( c_0 \), this alone suffices to ensure that \( f \in L^1(m_p) \), [Le, Theorem 5.1]. For \( p = 1 \), note that the constant function \( 1 = \hat{1} \in \mathcal{B}(\mathbb{T}) \) belongs to \( V^1(\mathbb{T}) \) and so \( \int_T |f| \hat{1} \ dt = \int_T |f| \hat{1} \ dt < \infty \), that is, \( f \in L^1(\mathbb{T}) = L^1(m_1) \); see Remark 2.6. \( \square \)

Fix \( 1 < p \leq 2 \) and let \( f \in \Phi^p(\mathbb{T}) \); see (1.5). That is, \( f \in L^1(\mathbb{T}) \) has the property that \( \hat{\chi}_A f \in \ell^p(\mathbb{Z}) \) for every \( A \in \mathcal{B}(\mathbb{T}) \). Then the set function \( \nu_f : \mathcal{B}(\mathbb{T}) \to \ell^p(\mathbb{Z}) \) defined by

\[
A \mapsto \nu_f(A) := \hat{\chi}_A f, \quad A \in \mathcal{B}(\mathbb{T}),
\]

is surely finitely additive. Actually more is true.

**Lemma 3.2.** Let \( 1 < p \leq 2 \). Then, for each \( f \in \Phi^p(\mathbb{T}) \), the finitely additive set function \( \nu_f \) as given by (3.4) is \( \sigma \)-additive, that is, it is an \( \ell^p(\mathbb{Z}) \)-valued vector measure on \( \mathcal{B}(\mathbb{T}) \).

**Proof.** Let \( \Gamma \) denote the linear span of \( \chi_m \), for \( m \in \mathbb{Z} \), and let \( \{A_n\}_{n \in \mathbb{N}} \) be a pairwise disjoint sequence of sets in \( \mathcal{B}(\mathbb{T}) \). Let \( \{A_{n_k}\}_{k \in \mathbb{N}} \) be any subsequence of \( \{A_n\}_{n \in \mathbb{N}} \). Then, with \( B = \bigcup_{k \in \mathbb{N}} A_{n_k} \), the dominated convergence theorem gives, for each \( m \in \mathbb{Z} \), that

\[
\sum_{k \in \mathbb{N}} \langle \nu_f(A_{n_k}), \chi_m \rangle = \sum_{k \in \mathbb{N}} \int_T f(t) \chi_{A_{n_k}}(t) e^{-int} dt = \int_T \hat{f}(m) = n_k \chi_{B}(m) = \sum_{k \in \mathbb{N}} \langle \nu_f(\bigcup_{k \in \mathbb{N}} A_{n_k}), \chi_m \rangle.
\]

(3.5)

So, every subseries of \( \sum_{k \in \mathbb{N}} \nu_f(A_n) \) is weakly \( \Gamma \)-convergent. Since the reflexive space \( \ell^p(\mathbb{Z}) \) cannot contain an isomorphic copy of \( \ell^\infty \) and \( \Gamma \) is a total subset of \( (\ell^p(\mathbb{Z}))' = \ell^p(\mathbb{Z}) \), it follows from the strengthened version of the Orlicz-Pettis theorem, [DU, p.23], that \( \nu_f(\bigcup_{n \in \mathbb{N}} A_n) \) is unconditionally norm convergent (to \( \nu_f(B) \)). Accordingly, \( \nu_f \) is \( \sigma \)-additive. \( \square \)

**Proposition 3.3.** Let \( 1 < p \leq 2 \). Then \( L^1(m_p) = \Phi^p(\mathbb{T}) \).

**Proof.** By Corollary 2.4 and (1.5) it is clear that \( L^1(m_p) \subset \Phi^p(\mathbb{T}) \). Conversely, suppose that \( f \in \Phi^p(\mathbb{T}) \). Given \( h \in V^p(\mathbb{T}) \), there is \( \phi \in \ell^p(\mathbb{Z}) \) such that \( \hat{h} = \phi \) and

\[
\langle m_p, \phi \rangle(A) = \int_A \hat{h} \ dt, \quad A \in \mathcal{B}(\mathbb{T});
\]

(3.6)

see the proof of Lemma 3.1. According to Lemma 3.2,

\[
A \mapsto \langle \nu_f(A), \phi \rangle = \langle f \hat{\chi}_A, \hat{h} \rangle, \quad A \in \mathcal{B}(\mathbb{T}),
\]

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is σ-additive. Define \( A_n = |f|^{-1}([0, n]) \) for \( n \in \mathbb{N} \), in which case \( A \cap A_n \uparrow A \) for each \( A \in \mathcal{B}(\mathbb{T}) \). By σ-additivity of \( \langle \nu_f(A), \phi \rangle \) we have

\[
\langle \nu_f(A), \phi \rangle = \lim_{n \to \infty} \langle f\chi_{A \cap A_n}, \tilde{h} \rangle.
\]

Since each function \( f\chi_{A \cap A_n} \) is bounded and \( h \in L^2(\mathbb{T}) \), Parseval’s formula gives

\[
\langle \nu_f(A), \phi \rangle = \lim_{n \to \infty} \int_T f\chi_{A \cap A_n} \tilde{h} \, d\mu = \lim_{n \to \infty} \int_A f \, d\mu,
\]

where the functions \( f_n = f\chi_{A_n} \in L^\infty(\mathbb{T}) \) converge pointwise to \( f \) on \( \mathbb{T} \) and \( d\mu = \tilde{h} \, d\mu \) is a complex measure. By [Le2, Lemma 2.3] we conclude that \( f \) is \( \mu \)-integrable (i.e. \( fh \in L^1(\mathbb{T}) \)) and

\[
\int_A f \, d\mu = \int_A f \, d\mu = \langle \nu_f(A), \phi \rangle.
\]

So, \( fh \in L^1(\mathbb{T}) \) for all \( h \in V^p(\mathbb{T}) \). Then Lemma 3.1 implies that \( f \in L^1(m_p) \).

We have an immediate consequence for the spaces \( \Gamma^p(\mathbb{T}) \) as given by (1.6).

**Corollary 3.4.** For each \( 1 < p \leq 2 \) we have \( L^1(m_p) = \Gamma^p(\mathbb{T}) \).

**Proof.** Let \( f \in \Gamma^p(\mathbb{T}) \). Then the operator \( S_f \) (see (1.3)) maps each \( h \in L^\infty(\mathbb{T}) \) into \( \ell^p'(\mathbb{Z}) \). In particular, for \( h = \chi_A \) we have

\[
S_f(\chi_A) = f\chi_A \in \ell^p'(\mathbb{Z}), \quad A \in \mathcal{B}(\mathbb{T}),
\]

that is, \( f \in \Phi^p(\mathbb{T}) \). By Proposition 3.3 we have \( f \in L^1(m_p) \).

Conversely, suppose that \( f \in L^1(m_p) \). Given \( h \in L^\infty(\mathbb{T}) \), we have a.e. \( |h| \leq \|h\|_\infty \chi_T \). Since the Lebesgue null sets and \( m_p \)-null sets coincide, we also have

\[
|h| \leq \|h\|_\infty \chi_T, \quad m_p - \text{a.e.} \tag{3.7}
\]

In particular, \( h \in L^\infty(m_p) \) and so \( hf \in L^1(m_p) \) by the ideal property of the Banach function space \( L^1(m_p) \). Then Corollary 2.4 can be applied to yield \( S_f(h) = \widehat{fh} = \int_T fh \, dm_p \in \ell^p'(\mathbb{Z}) \) and hence, that

\[
\|S_f(h)\|_{p'} = \|\int_T fh \, dm_p\|_{p'} \leq \|fh\|_{L^1(m_p)}.
\]

Since the norm of \( L^1(m_p) \) is a lattice norm, by (3.7) we get \( \|fh\|_{L^1(m_p)} \leq \|h\|_\infty \|f\|_{L^1(\mathbb{T})} \). Accordingly,

\[
\|S_f(h)\|_{p'} \leq \|h\|_\infty \|f\|_{L^1(m_p)}.
\]

This shows that \( S_f \) is a bounded operator from \( L^\infty(\mathbb{T}) \) to \( \ell^p'(\mathbb{Z}) \) with \( \|S_f\|_{\infty,p'} \leq \|f\|_{L^1(m_p)} \). In particular, \( f \in \Gamma^p(\mathbb{T}) \). \( \square \)
Proof of Theorem 1.2. Since \( F^p(\mathbb{T}) = L^1(m_p) \), it follows from Lemma 3.1 that \( F^p(\mathbb{T}) = \Delta^p(\mathbb{T}) \), for all \( 1 \leq p \leq 2 \). It is clear from (1.5) and (1.6) that \( \Phi^1(\mathbb{T}) = L^1(\mathbb{T}) \) and hence, \( \Phi^1(\mathbb{T}) = \Gamma^1(\mathbb{T}) = F^1(\mathbb{T}) \) by Remark 2.6. For \( 1 < p \leq 2 \), it follows from Proposition 3.3 that \( F^p(\mathbb{T}) = \Phi^p(\mathbb{T}) \) and from Corollary 3.4 that \( F^p(\mathbb{T}) = \Gamma^p(\mathbb{T}) \). Moreover, by (2.7) and Corollary 2.4 we have

\[
\| f \|_{L^1(m_p)} \leq 4 \sup_{A \in B(\mathbb{T})} \| \int_{\mathbb{T}} \chi_A f \ dm_p \|_{\ell^p} = 4 \sup_{A \in B(\mathbb{T})} \| S_f(\chi_A) \|_{\ell^p} \leq 4 \| S_f \|_{\ell^\infty,\ell^p}.
\]

Accordingly, for \( f \in L^1(m_p) = \Gamma^p(\mathbb{T}) \) the norms \( \| f \|_{L^1(m_p)} \) and \( \| S_f \|_{\ell^\infty,\ell^p} \) are equivalent. □

In the remainder of this section we consider some properties of the optimal lattice domain \( F^p(\mathbb{T}) = L^1(m_p) \).

Fix \( 1 < p < 2 \). The **associate space** of the Banach function space \( F^p(\mathbb{T}) \) consists of all \( h \in L^0(\mathbb{T}) \) satisfying

\[
\int_{\mathbb{T}} |fh| \ dt < \infty, \quad f \in F^p(\mathbb{T}),
\]

equipped with the norm \( \sup \{ \int_{\mathbb{T}} |fh| \ dt : \| f \|_{F^p(\mathbb{T})} = 1 \} \), [Za, Ch.15, Sect. 69]. Since \( F^p(\mathbb{T}) \) has \( \sigma \)-order continuous norm, the Banach space dual of \( F^p(\mathbb{T}) \) coincides with its associate space, [Za, p.480]. Moreover, \( (F^p(\mathbb{T}))' \) is again a Banach function space in \( L^0(\mathbb{T}) \), [Za, p.457]. In particular, it is an ideal in \( L^0(\mathbb{T}) \). As noted in the proof of Lemma 3.1, a function \( f \in L^0(\mathbb{T}) \) belongs to \( L^1(m_p) \) if and only if it satisfies (2.3). This implies that \( L^1(m_p) = F^p(\mathbb{T}) \) is weakly sequentially complete, has the \( \sigma \)-Falou property (i.e. \( 0 \leq f_n \uparrow f \) with \( \{ f_n \} \subset F^p(\mathbb{T}) \) a norm bounded sequence implies that \( \| f_n \|_{F^p(\mathbb{T})} \uparrow \| f \|_{F^p(\mathbb{T})} \) and that \( L^1(m_p) \) coincides with its second associate space, [CR4, Prop. 2.1, 2.3, 2.4].

Note that the operator norm of \( S_f \in L(\ell^\infty(\mathbb{T}), \ell^p(\mathbb{Z})) \), for \( f \in F^p(\mathbb{T}) \), agrees with the norm of the dual operator

\[
S'_f : \ell^p(\mathbb{Z}) \ni \{ a_n \} \mapsto f(x) \sum_{n \in \mathbb{Z}} a_n e^{-inx} \in L^1(\mathbb{T}) \subset L^\infty(\mathbb{T})'.
\]

Since \( \ell^p(\mathbb{Z}) \) is modulation invariant, i.e. \( \{ a_n \} \) and \( \{ e^{inx} a_n \} \) have the same norm, it is clear that \( F^p(\mathbb{T}) \) is translation invariant. Moreover, it is easy to check that the translation operators \( \tau_t f(x) = f(x - t) \) are continuous in \( F^p(\mathbb{T}) \) and that \( \tau_t \) converges to the identity for the strong operator topology as \( t \to 0 \). Accordingly, \( F^p(\mathbb{T}) \) is a **homogeneous Banach space**, [Ka].

If \( h \in V^p(\mathbb{T}) \), then (3.8) holds because of (1.4) and Theorem 1.2. From the natural inclusion \( L^p(\mathbb{T}) \subset F^p(\mathbb{T}) \) we then conclude that

\[
V^p(\mathbb{T}) \subset (F^p(\mathbb{T}))' \subset L^\prime(\mathbb{T}). \tag{3.9}
\]

Actually, since \( \chi_T \in V^p(\mathbb{T}) \) and \( (F^p(\mathbb{T}))' \) is an ideal, we see that also \( L^\infty(\mathbb{T}) \subset (F^p(\mathbb{T}))' \). It follows easily from (3.8) that \( (F^p(\mathbb{T}))' \) is translation invariant. According to [Ka, IV Theorem 2.4], there exists \( h \in C(\mathbb{T}) \) such that \( \check{h} \notin \ell^p(\mathbb{Z}) \). If the first containment in (3.9) was an equality,
then \( V^p(\mathbb{T}) \) would be an ideal and so the inequality \( |h| \leq \|h\|_\infty \chi_\mathbb{T} \) would give that \( h \in V^p(\mathbb{T}) \), which is not the case. Of course, since \( V^p(\mathbb{T}) \) contains the trigonometric polynomials, it surely separates the points of \( F^p(\mathbb{T}) \). If the second containment in \((3.9)\) was an equality, then \( L^p(\mathbb{T}) \) would coincide with the second associate space of \( L^1(m_p) \) which, as noted above, equals \( L^1(m_p) = F^p(\mathbb{T}) \). This contradicts Theorem 1.4. So, both containments in \((3.9)\) are proper.

From the viewpoint of analysis, the weak sequential completeness of \( F^p(\mathbb{T}) \) is difficult to use in practice since \((F^p(\mathbb{T}))'\) is not explicitly known. However, there is available a good substitute in this regard. Indeed, Theorem 1.1 (iii) and the \( \sigma \)-Fatou property of \( F^p(\mathbb{T}) \) show that \( F^p(\mathbb{T}) \) is also a Banach function space in the more restricted sense of \([BS]\). Since \( L^\infty(\mathbb{T}) \) is an order ideal of \((F^p(\mathbb{T}))'\) containing the simple functions, it follows from \([BS, \text{Ch.1}, \text{Theorem 5.2}]\) that \( F^p(\mathbb{T}) \) is also sequentially \( \sigma(F^p(\mathbb{T}), L^\infty(\mathbb{T})) \)-complete.

### 4 Proof of Theorem 1.4

The proof of Theorem 1.4, for \( p' > 2 \) an even integer, is somewhat easier because in this case we can rely on the Hardy-Littlewood majorant property of the spaces \( L^p(\mathbb{T}), [HL] \). To see this and to get an idea of what type of functions are contained in \( F^p(\mathbb{T}) \) we establish, e.g., for \( p = 4/3 \), the following

**Lemma 4.1.** If \( f \in L^1(\mathbb{T}) \) is non-negative and \( \hat{f} \in \ell^4(\mathbb{Z}) \), then \( f \in F^{4/3}(\mathbb{T}) \) and

\[
\|f\|_{F^{4/3}(\mathbb{T})} \leq 4 \|\hat{f}\|_{\ell^4(\mathbb{Z})}.
\]

**Proof.** This follows from Parsevals’s identity as follows. For \( g \in L^\infty(\mathbb{T}) \) we have

\[
\|\hat{f}g\|_4^4 = \sum_{n \in \mathbb{Z}} |\hat{f}g(n)\hat{f}g(n)|^2 = \int_T |fg \ast fg|^2 \, dt \leq \|g\|_{4}^4 \int_T |f \ast f|^2 \, dt = \|g\|_{\infty}^4 \|\hat{f}\|_4^4,
\]

that is, \( \|S_f\|_{\infty,4} \leq \|\hat{f}\|_4 \). Hence, \( \|f\|_{F^{4/3}(\mathbb{T})} \leq 4 \|S_f\|_{\infty,4} \leq 4 \|\hat{f}\|_4. \)

What we have said so far applies also for higher dimensional tori \( \mathbb{T}^d \cong (-\pi, \pi]^d \). In particular, we may apply the previous Lemma to \( \mathbb{T}^2 \) to see that \( L^{4/3}(\mathbb{T}^2) \) is a proper subspace of \( F^{4/3}(\mathbb{T}^2) \). In fact, for \( \alpha > 0 \), the Fourier transform of the non-negative function \( M_\alpha : x \mapsto \frac{1}{\Gamma(\alpha)} (1 - |x|^2)^{\alpha-1}, \) defined on \( (-\pi, \pi]^2 \), decays asymptotically as \( |n|^{-\frac{3}{2} + \alpha} \) for \( n \to \infty \) in \( \mathbb{Z}^2 \). Therefore, \( M_\alpha \in F^{4/3}(\mathbb{T}^2) \) for all \( \alpha > 0 \), whereas \( M_\alpha \) is obviously not an \( L^{4/3}(\mathbb{T}^2) \)-function for \( \alpha \leq 1/4 \).

We note that for \( \alpha \to 0 \) the functions \( M_\alpha \), considered as distributions on \( \mathbb{T}^2 \), converge to arclength measure \( d\sigma \) on the circle \( S^1 \). Hence, for \( \alpha \to 0 \), the \( L^1 \)-function \( M_\alpha \) does not converge in the \( F^{4/3}(\mathbb{T}^2) \)-norm. However, it was shown by E.M. Stein (see \([Fe], [St]\)) that the operator

\[
S_\sigma(f) = \hat{f}d\sigma, \quad f \in C^\infty(S^1),
\]
maps $L^2(S^1, d\sigma)$ and hence, also $L^\infty(S^1, d\sigma)$, boundedly into $L^q(\mathbb{R}^2)$ for some $q > 4$. The fact that $S_\alpha$ maps $L^\infty(S^1, d\sigma)$ into $L^4(\mathbb{R}^2)$ for all $q > 4$ was also shown in [Fe]; see also [St]. An easy argument shows that we may replace $L^q(\mathbb{R}^2)$ with $\ell^q(\mathbb{Z}^2)$.

This motivates us to use Fourier restriction theory to establish that the inclusion $L^p(\mathbb{T}) \subset F^p(\mathbb{T})$ is proper. For the proof of Theorem 1.4 we will employ the following result for Salem measures, [Mo, Mo2, Sa].

**Proposition 4.2.** There is a non-negative measure $\mu$ on $\mathbb{R}$ with the following properties.

(i) $E = \text{supp}(\mu)$ is a compact subset of $[-1, 1] \subset (-\pi, \pi]$ of Hausdorff dimension $\alpha \in (0, 1)$.

(ii) There is $C > 0$ such that, for each interval $I \subset \mathbb{R}$, we have

$$\mu(I) \leq C |I|^{\alpha}.$$  

(iii) For each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that the Fourier transform of $\mu$ (on $\mathbb{R}$) satisfies the asymptotic bound

$$|\hat{\mu}(\xi)| \leq C_\varepsilon |\xi|^{-\frac{\alpha}{2} + \varepsilon}, \quad |\xi| \to \infty.$$  

(iv) The following analogue of the Stein-Tomas restriction inequality holds:

$$\int |\hat{f}(y)|^2 d\mu(y) \leq C \|f\|_{L^p(\mathbb{R})}^2, \quad f \in L^p(\mathbb{R}),$$  

for $1 \leq p < p_\varepsilon(\alpha)$, where $p_\varepsilon(\alpha) \to \frac{2(2-\alpha)}{4-3\alpha}$ as $\varepsilon \to 0^+$. \hfill $\Box$

We will need to transfer inequality (4.1) to the torus. For a sequence $\{f_m\} \in \ell^p(\mathbb{Z})$ we consider $f(x) = \sum_{m \in \mathbb{Z}} f_m \chi_Q(2\pi m + x)$, where $Q = (-\pi, \pi]$ is a fundamental interval for the lattice $2\pi \mathbb{Z}$. Obviously we have $f \in L^p(\mathbb{R})$. Now apply (4.1) to $f$. Since $\hat{f}(\xi) = \hat{\chi}_Q(\xi) \sum_{m \in \mathbb{Z}} f_me^{im\xi} =: \hat{\chi}_Q(\xi) F(\xi)$ and $|\chi_Q(\xi)| > 1/2$ on $E$, we obtain for the periodic function $F$ the inequality

$$\int |F|^2 d\mu \leq C \left( \sum_{m \in \mathbb{Z}} |f_m|^p \right)^{2/p}.$$  

Hence, for each $g \in L^\infty(E, d\mu)$, we get by the dual inequality of (4.2) that

$$\left( \sum_{m \in \mathbb{Z}} |gd\mu(m)|^p \right)^{1/p} \leq C \|g\|_{L^2(E, d\mu)} \leq C \|g\|_{L^\infty(E, d\mu)}.$$  

Denote by $\mu_t$ the translation of $\mu$ by $t \in \mathbb{R}$, let $I$ be an open interval centred at 0 of length $1/10$, let $\phi \in C^\infty(I)$ be non-negative with $\phi(0) = 1 = \hat{\phi}(0)$ and, for $0 < \beta < 1$, define $r_\beta(t) = |t|^{-\beta} \phi(t)$. Now define the non-negative function

$$I_\beta(y) = \int_{\mathbb{R}} r_\beta(t) \, d\mu_1(y) = (r_\beta * \mu)(y), \quad y \in \mathbb{R}.$$  

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Note that $I_\beta \in L^1(\mathbb{R})$ and $\text{supp } I_\beta$ is a proper subset of $Q$. Clearly, the left-hand side of (4.1) is translation invariant and so

$$\int |\hat{f}(y)|^2 \, d\mu(y) \leq C \|f\|^2_{L^p(\mathbb{R})}, \quad t \in \mathbb{R}.$$ 

Multiplying by $r_\beta$ and then integrating with respect to $t$, gives

$$\int |\hat{f}(y)|^2 \, I_\beta(y) \, dy \leq C \|f\|^2_{L^p(\mathbb{R})}.$$ 

As above we obtain, for $F(x) = \sum_{m \in \mathbb{Z}} f_m e^{imx}$, that

$$\int_Q |F|^2 \, I_\beta(y) \, dy \leq C \left( \sum_{m \in \mathbb{Z}} |f_m|^p \right)^{2/p}$$ 

and therefore, for each $g \in L^\infty(\mathbb{T})$, that

$$\left( \sum_{m \in \mathbb{Z}} |g I_\beta(m)|^p \right)^{1/p} \leq C \|g\|_{L^2(Q,I_\beta(y))} \leq C \|g\|_{L^\infty(Q)} \int_Q I_\beta(y) \, dy.$$ 

That is, $I_\beta \in \mathcal{F}^p(\mathbb{T})$ for $0 < \beta < 1$ and $1 \leq p < p_c(\alpha)$.

**Proof of Theorem 1.4.** We will show, for an appropriate choice of $\alpha$ and $\beta$, that $I_\beta \notin L^p(Q)$. Note that:

- from Proposition 4.2 (ii) we obtain $I_\beta \in L^\infty(Q)$ for $\beta < \alpha$ and, of course, $I_\beta \in L^1(Q)$ for $0 < \beta < 1$. Therefore, by convexity we get, for $\beta > \alpha$, that

$$I_\beta \in L^p(Q), \quad \text{if } p < \frac{1 - \alpha}{\beta - \alpha}. \quad (4.5)$$

Below we will see that this condition is essentially sharp.

- Since $|\hat{r}_\beta(\xi)| \geq c \, |\xi|^{-(1-\beta)}$ as $|\xi| \to \infty$, we obtain

$$|\hat{I}_\beta(\xi)| = |\hat{\mu}(\xi) \hat{r}_\beta(\xi)| \geq c \, |\hat{\mu}(\xi)| \, |\xi|^{-(1-\beta)}, \quad |\xi| \to \infty.$$ 

Therefore

$$\int |\hat{I}_\beta(\xi)|^2 \, d\xi \geq c \int |\hat{\mu}(\xi)|^2 \, (1 + |\xi|)^{-2(1-\beta)} \, d\xi \approx I_t(d\mu),$$

where $t = 2\beta - 1$ and $I_t(d\mu)$ is the $t$-energy of $\mu$ (see [Fa]). From property (iii) in Proposition 4.2 we obtain that $t \leq \dim_H E = \alpha$ (see [Fa, p.79]). That is, $\beta \leq \frac{1+\alpha}{2}$ provided $\int_{\mathbb{R}} |\hat{I}_\beta(\xi)|^2 \, d\xi$ is finite.
Suppose now that $\beta_0 > \alpha$ and $I_{\beta_0} \in L^{q_0}(Q)$ for $q_0 := \frac{4-\alpha + \delta}{\beta_0 - \alpha}$ and some $\delta > 0$. Since $I_\beta \in L^\infty(Q)$ for $\beta < \alpha$, by convexity we obtain that $I_\beta \in L^2(\mathbb{R})$ for all $\beta < \frac{1+\alpha}{2} + \frac{\delta}{2}$. Hence, $\widehat{I_\beta} \in L^2(\mathbb{R})$, that is, the $t$-energy of $\mu$ is finite for all $t = 2\beta - 1 < \alpha + \delta$. Accordingly, $\delta = 0$.

Hence, for a given $p \in (1, 2)$ we may choose $\alpha \in (0, 1)$ such that $\frac{2(2-\alpha)}{4-3\alpha} > p$. By choosing $\beta > \alpha$ sufficiently close to 1 we can ensure that $I_\beta \notin L^p(T)$, but $I_\beta \in F^p(T)$. □

We conclude with the observation that $L^r(T) \not\subseteq F^p(T)$ for $1 \leq r < p$ and $F^p(T) \not\subseteq L^r(T)$ for $1 < r \leq p$. The first statement follows by considering $f(t) = |t|^{-1/p}$. On the other hand, the above construction ensures, for any $r \in (1, p)$, that the space $F^p(T)$ is not contained in $L^r(T)$. This is not surprising, since $L^p$-spaces merely measure a local property, whereas the $F^p(T)$-norm involves not only local properties but also arithmetic properties of a function (e.g. in case of $I_\beta$, ”by lack of a better description” this means, not only are its peaks important but also how they are distributed relative to each other). One may also see this by estimating the $F^p(T)$-norm of $f(mx)$ for $m \in \mathbb{N}$ and $f \in F^p(T)$.

References


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