Optimal extension of the Hausdorff-Young inequality

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Abstract

Given $1 < p < 2$, we construct a Banach function space $\mathbf{F}^p(\mathbb{T})$ with σ-order continuous norm which contains $L^p(\mathbb{T})$ and has the property that the Fourier transform map $F : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ has a continuous $\ell^{p'}(\mathbb{Z})$-valued extension to $\mathbf{F}^p(\mathbb{T})$. Moreover, $\mathbf{F}^p(\mathbb{T})$ is maximal with these properties and satisfies $L^p(\mathbb{T}) \subset \mathbf{F}^p(\mathbb{T}) \subset L^1(\mathbb{T})$ with both containments proper. Each $\mathbf{F}^p(\mathbb{T})$ turns out to be a weakly sequentially complete, translation invariant, homogeneous Banach space and consists precisely of those functions $f \in L^1(\mathbb{T})$ such that $\widehat{f\chi_A} \in \ell^{p'}(\mathbb{Z})$ for every Borel set $A \subset \mathbb{T}$. This answers a question of R.E. Edwards posed some 40 years ago.

1 Introduction and main results

It is known that for $1 \leq p \leq 2$ the Fourier transform F maps $L^p(\mathbb{T})$ into $\ell^{p'}(\mathbb{Z})$, where $1/p' + 1/p = 1$, and the Hausdorff-Young inequality

$$\|\hat{f}\|_{p'} \leq \|f\|_p, \qquad f \in L^p(\mathbb{T})$$

ensures that F is continuous. Moreover, the Fourier transform is an injective map from distributions $D(\mathbb{T})$ into the space of sequences of polynomial growth. The theme of this note is to address the following question: Is the Hausdorff-Young inequality optimal? That is, keeping the range space $\ell^{p'}(\mathbb{Z})$ fixed, is it possible to continuously extend the Fourier transform operator $F : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ to a Banach function space $\mathbf{F}^p(\mathbb{T})$, over the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$ (see Section 2 for the definition), which is larger than $L^p(\mathbb{T})$ and in such a way that $\mathbf{F}^p(\mathbb{T})$ is maximal (or optimal) with this property? Moreover, if so, can $\mathbf{F}^p(\mathbb{T})$ be identified in some "concrete" way? That there exist distributions which are not in $L^p(\mathbb{T})$, but whose Fourier transform lies in $\ell^{p'}(\mathbb{Z})$, is known. Here are some examples (see [St, p.339],[Zyg, II, p.*102*]). For almost all sign changes $\sum_{n\geq 1} \pm n^{-1/2} \cos nx$ is not integrable while its Fourier transform is in all $\ell^{p'}(\mathbb{Z})$ for $p' > 2$. An even more concrete example is a function with a Riemann singularity of order $0 < \lambda < 2$ at 0, say

$$f_\lambda(x) = e^{i/x}\, x^{-\lambda}, \qquad 0 < x < 1 \tag{1.1}$$

and $f_\lambda(x) = 0$ elsewhere in $(-\pi, \pi]$, which has Fourier transform $\hat{f}_\lambda(n) = \sqrt{i\pi}\, e^{2i\sqrt{n}}\, n^{-3/4+\lambda/2} + O(n^{-1+\lambda/2})$ if $n \to +\infty$ and decays like the power given in the O-term for $n \to -\infty$. Hence,

$\hat{f}_\lambda$ lies in $\ell^{p'}(\mathbb{Z})$ for certain $p' > 2$ depending on λ while, for $\lambda > 1$, the function $f_\lambda \in L^0(\mathbb{T})$ is not integrable at 0. Here, $L^0(\mathbb{T})$ denotes the space of all complex-valued $\mathcal{B}(\mathbb{T})$-measurable (i.e. Borel measurable) functions on $\mathbb{T}$.

Theorem 1.1. *Let $1 \leq p \leq 2$. There exists a Banach function space $\mathbf{F}^p(\mathbb{T}) \subset L^0(\mathbb{T})$ with σ-order continuous norm $\|\cdot\|_{\mathbf{F}^p(\mathbb{T})}$ and having the following properties:*

(i) $L^p(\mathbb{T})$ is continuously included in $\mathbf{F}^p(\mathbb{T})$ and the Fourier transform map $F : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ has an extension to a continuous linear operator from $\mathbf{F}^p(\mathbb{T})$ into $\ell^{p'}(\mathbb{Z})$. More precisely,

$$\|f\|_{\mathbf{F}^p(\mathbb{T})} \leq 4\,\|f\|_p, \qquad f \in L^p(\mathbb{T}).$$

(ii) If Z is any Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$ with σ-order continuous norm such that $L^p(\mathbb{T})$ is continuously included in Z and F has an extension to a continuous linear operator from Z into $\ell^{p'}(\mathbb{Z})$, then Z is continuously included in $\mathbf{F}^p(\mathbb{T})$.

(iii) $\mathbf{F}^p(\mathbb{T}) \subset L^1(\mathbb{T})$ with $\|f\|_1 \leq \|f\|_{\mathbf{F}^p(\mathbb{T})}$. Moreover, the $\ell^{p'}(\mathbb{Z})$-valued extension of F from $L^p(\mathbb{T})$ to $\mathbf{F}^p(\mathbb{T})$ is again the map $f \to \hat{f}$ for $f \in \mathbf{F}^p(\mathbb{T})$.

Theorem 1.1 justifies the statement (within a well defined and extensive class of spaces) that the Hausdorff-Young inequality has an $\ell^{p'}(\mathbb{Z})$-valued extension to a larger maximal domain $\mathbf{F}^p(\mathbb{T})$, which we will call its *optimal lattice domain*. By (ii), $\mathbf{F}^p(\mathbb{T})$ is unique up to isomorphism; its norm turns out to be

$$\|f\|_{\mathbf{F}^p(\mathbb{T})} = \sup\{\int_{\mathbb{T}} |f|\; |\check{\phi}|\; dt \;:\; \phi \in \ell^p(\mathbb{Z}),\; \|\phi\|_p = 1\}.$$

According to (iii), neither the random series mentioned above nor the functions (1.1) with Riemann singularities for $1 < \lambda < 2$ are contained in $\mathbf{F}^p(\mathbb{T})$. We remark that the above mentioned result and those below are also valid for higher dimensional tori $\mathbb{T}^d, d > 1$. It will become apparent in the sequel that the restriction conjecture for the Fourier transform (see [St2]) can be rephrased as finding sharp bounds for the $\mathbf{F}^p(\mathbb{T}^d)$-norm of smooth bump functions with support in a δ-neighbourhood of the d-dimensional unit sphere.

We now turn to more concrete descriptions of $\mathbf{F}^p(\mathbb{T})$. Given $1 \leq p \leq 2$, define a vector subspace $V^p(\mathbb{T})$ of $L^{p'}(\mathbb{T})$ by

$$V^p(\mathbb{T}) = \{h \in L^{p'}(\mathbb{T}) \;:\; h = \check{\phi} \text{ for some } \phi \in \ell^p(\mathbb{Z})\}. \tag{1.2}$$

For each $f \in L^1(\mathbb{T})$ define a linear map $S_f : L^\infty(\mathbb{T}) \to c_0(\mathbb{Z})$ by

$$S_f : g \mapsto \widehat{fg}, \qquad g \in L^\infty(\mathbb{T}). \tag{1.3}$$

Clearly S_f is continuous with operator norm $\|S_f\| \leq \|f\|_1$. For each $1 \leq p \leq 2$, let $\mathcal{L}(L^\infty(\mathbb{T}), \ell^{p'}(\mathbb{Z}))$ denote the Banach space of all continuous operators $T : L^\infty(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ equipped with the operator norm

$$\|T\|_{\infty,p'} = \sup_{\|g\|_\infty = 1} \|Tg\|_{p'}.$$

If $f \in L^1(\mathbb{T})$ has the property that $Range(S_f) \subset \ell^{p'}(\mathbb{Z})$, then the closed graph theorem implies that $\|S_f\|_{\infty,p'} < \infty$.

Theorem 1.2. *Let $1 \le p \le 2$. Each of the spaces*

$$\begin{aligned} \Delta^p(\mathbb{T}) &= \{f \in L^1(\mathbb{T}) : \int_{\mathbb{T}} |fg| \, dt < \infty, \forall g \in V^p(\mathbb{T})\}, & (1.4) \\ \Phi^p(\mathbb{T}) &= \{f \in L^1(\mathbb{T}) : \widehat{f\chi_A} \in \ell^{p'}(\mathbb{Z}), \forall A \in \mathcal{B}(\mathbb{T})\}, & (1.5) \\ \Gamma^p(\mathbb{T}) &= \{f \in L^1(\mathbb{T}) : Range(S_f) \subset \ell^{p'}(\mathbb{Z})\}, & (1.6) \end{aligned}$$

coincides with the optimal lattice domain $\mathbf{F}^p(\mathbb{T})$ of the Hausdorff-Young inequality. Moreover, in the case of (1.6), the operator norm $\|S_f\|_{\infty,p'}$ is equivalent to the norm of f in $\mathbf{F}^p(\mathbb{T})$, for each $f \in \mathbf{F}^p(\mathbb{T})$.

Remark 1.3. (i) For $p = 1$ it turns out that $\mathbf{F}^1(\mathbb{T}) = L^1(\mathbb{T})$ and for $p = 2$ that $\mathbf{F}^2(\mathbb{T}) = L^2(\mathbb{T})$. So, both the Fourier transform maps $F : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ and $F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ are already defined on their optimal domain. Also, for $1 \le p < q \le 2$ we have $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$ and therefore $V^p(\mathbb{T}) \subset V^q(\mathbb{T}) \subset L^2(\mathbb{T})$. It is then clear from (1.4) that $L^2(\mathbb{T}) \subset \mathbf{F}^q(\mathbb{T}) \subset \mathbf{F}^p(\mathbb{T}) \subset L^1(\mathbb{T})$.

(ii) It is not obvious from (1.5) that the space $\Phi^p(\mathbb{T})$ is actually an ideal relative to the pointwise a.e. order in $L^0(\mathbb{T})$. That is, if $f \in \Phi^p(\mathbb{T})$ and $g \in L^0(\mathbb{T})$ satisfies $|g| \le |f|$ a.e., then also $g \in \Phi^p(\mathbb{T})$. Of course, being equal to the Banach function space $\mathbf{F}^p(\mathbb{T})$, it must have this property. In addition to having σ-order continuous norm, it will be seen that the optimal domain $\mathbf{F}^p(\mathbb{T})$ has other desirable properties; it is translation invariant, weakly sequentially complete, has the σ-Fatou property, etc. (see the end of Section 3).

(iii) For $1 \le p \le 2$, the following question was raised some forty years ago by R.E. Edwards,[Ed, p.206]; *What can be said about the family of functions $f \in L^1(\mathbb{T})$ having the property that $\widehat{f\chi_A}$ lies in $\ell^{p'}(\mathbb{Z})$ for all $A \in \mathcal{B}(\mathbb{T})$?* Theorems 1.1 and 1.2 provide an exact answer: this family of functions is precisely the optimal lattice domain $\mathbf{F}^p(\mathbb{T})$ of the Fourier transform map $F : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$.

Remark 1.3(iii) raises the question of whether $\mathbf{F}^p(\mathbb{T})$ is genuinely larger than $L^p(\mathbb{T})$.

Theorem 1.4. *Let $1 < p < 2$. Then both the inclusions $L^p(\mathbb{T}) \subset \mathbf{F}^p(\mathbb{T}) \subset L^1(\mathbb{T})$ are proper.*

Remark 1.5. It is known that there exists $f \in L^1(\mathbb{T})$ whose Fourier transform does not lie in $\ell^{p'}(\mathbb{Z})$ for any $p < \infty$, e.g. $f(t) = \sum_{n=2}^{\infty} \frac{\cos nt}{\log n}$ has this property. Accordingly, the inclusion $\mathbf{F}^p(\mathbb{T}) \subset L^1(\mathbb{T})$ is always proper. That the other inclusion is also proper will be established in Section 4.

What is the connection between the (apparently) abstract space $\mathbf{F}^p(\mathbb{T})$ in the statement of Theorem 1.1 with the more concrete descriptions of $\mathbf{F}^p(\mathbb{T})$ given in Theorem 1.2? It is routine to check that the set function $m_p : \mathcal{B}(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ defined by

$$m_p : A \mapsto F(\chi_A), \quad A \in \mathcal{B}(\mathbb{T}), \tag{1.7}$$

is σ-additive, that is, it is an $\ell^{p'}(\mathbb{Z})$-valued vector measure. Moreover, the m_p-null sets coincide with the Lebesgue null sets in $\mathbb{T}$. This crucial point allows us to view the Banach lattice $L^1(m_p)$ of all m_p-integrable functions (modulo m_p-null functions) as a Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$. It is precisely the space $L^1(m_p)$ which is proved to have the optimality properties required of $\mathbf{F}^p(\mathbb{T})$ in Theorem 1.1. That is, $\mathbf{F}^p(\mathbb{T}) = L^1(m_p)$ and the integration map $f \mapsto \int_{\mathbb{T}} f dm_p$, from $L^1(m_p)$ to $\ell^{p'}(\mathbb{Z})$, is precisely the continuous extension of F from $L^p(\mathbb{T})$ to $\mathbf{F}^p(\mathbb{T})$. This approach to optimal extensions, via the integration map of appropriate vector measures, has proved to be effective in recent years in the treatment of various operators/inequalities arising in classical analysis; see for example [CR1],[CR2],[CR3],[CR4],[OR1],[OR2] and the references therein. For a different extension of the Fourier transform we refer to [Gu] and the references therein.

2 Proof of Theorem 1.1

We begin with some preliminaries concerning integration with respect to a general vector measure. A set function $m : \Sigma \to X$, where X is a complex Banach space and Σ is a σ-algebra of subsets of a non-empty set Ω, is called a *vector measure* if it is σ-additive, that is, $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ for all pairwise disjoint sequences $\{A_n\}_{n=1}^{\infty}$ in Σ with the series being (norm) unconditionally convergent in X. A set $A \in \Sigma$ is called m-null if $m(B) = 0$ for all $B \in \Sigma$ which are contained in A. The variation $|m|$ of m is the smallest, $[0, \infty]$-valued measure satisfying $\|m(A)\| \le |m|(A)$, for all $A \in \Sigma$, and can be defined (as for scalar measures) via finite partitions, [DU, Ch.I]. If $|m|(\Omega) < \infty$, then m is said to have finite variation. The semi-variation of m is the set function $\|m\| : \Sigma \to [0, \infty)$ defined by

$$\|m\|(A) := \sup_{x' \in X', \|x'\|=1} |\langle m, x' \rangle|(A), \quad A \in \Sigma, \tag{2.1}$$

where X' is the dual Banach space of X and $\langle m, x \rangle$ denotes the complex measure $A \mapsto \langle m(A), x' \rangle$ on Σ, for each $x' \in X'$. Then

$$\sup_{B \in \Sigma, B \subset A} \|m(B)\| \le \|m\|(A) \le 4 \sup_{B \in \Sigma, B \subset A} \|m(B)\| \tag{2.2}$$

for each $A \in \Sigma$, [DU, p.4]. The vector measure m is said to have relatively compact range if the closure of its range $m(\Sigma) := \{m(A) : A \in \Sigma\}$ is a compact subset of X.

A Σ-measurable function $f : \Omega \to \mathbb{C}$ is called m-integrable if

$$\int_{\Omega} |f| \, d|\langle m, x' \rangle| < \infty, \quad x' \in X', \tag{2.3}$$

and for each $A \in \Sigma$ there exists a vector in X, necessarily unique and denoted by $\int_A f \, dm$, such that

$$\langle \int_A f dm, x' \rangle = \int_A f \, d\langle m, x' \rangle, \quad x' \in X'. \tag{2.4}$$

By the Orlicz-Pettis theorem, [DU, p.22], the set function

$$m_f : A \mapsto \int_A f dm, \quad A \in \Sigma, \tag{2.5}$$

is also an X-valued vector measure. The linear space of all m-integrable functions is denoted by $L^1(m)$; it is equipped with the lattice seminorm

$$\|f\|_{L^1(m)} := \sup_{x' \in X',\, \|x'\|=1} \int_\Omega |f| \; d|\langle m, x'\rangle|. \tag{2.6}$$

Note that $\|f\|_{L^1(m)} = \|m_f\|(\Omega)$, where $\|m_f\|(\cdot)$ is the semi-variation of the vector measure m_f. It follows from (2.2) applied to m_f that

$$\sup_{A\in\Sigma} \| \int_A f \; dm\| \le \|f\|_{L^1(m)} \le 4 \sup_{A\in\Sigma} \| \int_A f \; dm\| \tag{2.7}$$

for every $f \in L^1(m)$. A function $f \in L^1(m)$ is called m-null if $\|f\|_{L^1(m)} = 0$ or equivalently, if m_f is the zero vector measure. The quotient space of $L^1(m)$ modulo m-null functions is again identified with (and denoted by) $L^1(m)$. Then $L^1(m)$ is complete (i.e. a Banach space), [FNR], and the $\mathbb{C}$-valued, Σ-simple functions are dense in $L^1(m)$, [Le, Theorem 3.5]. Moreover, $L^1(m)$ is a complex Banach lattice relative to the pointwise m-a.e. order on Ω and the norm given by (2.6). That is, $|f| \le |g|$ m-a.e. implies $\|f\|_{L^1(m)} \le \|g\|_{L^1(m)}$. Moreover, the norm is σ-order continuous (as a consequence) of the dominated convergence theorem, [Le2, Theorem 2.2], meaning that if non-negative functions f_n decrease to 0 as $n \to \infty$ in the order of $L^1(m)$, then $f_n \to 0$ in $L^1(m)$ as $n \to \infty$. For these claims we refer to [FNR] and the references therein. Moreover, every Σ-measurable function $f : \Omega \to \mathbb{C}$ which satisfies $|f| \le K$, m-a.e., for some $K > 0$ (that is $f \in L^\infty(m)$), is necessarily m-integrable, [Le2, Theorem 2.2], and satisfies (via (2.6))

$$\|f\|_{L^1(m)} \le \|f\|_{L^\infty(m)} \; \|m\|(\Omega). \tag{2.8}$$

It follows from (2.7) that the integration map $I_m : f \mapsto \int_\Omega f \; dm$ is a continuous linear operator from $L^1(m)$ into X with operator norm $\|I_m\| = 1$.

It is time to specialize to the particular vector measure (1.7).

Lemma 2.1. *Let $1 \le p \le 2$.*

(i) The vector measure $m_p : \mathcal{B}(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ as given by (1.7) is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ (i.e. m_p-null sets are Lebesgue null sets).

(ii) For $1 < p \le 2$, m_p has infinite variation.

(iii) The containment $L^1(m_p) \subset L^1(\mathbb{T})$ is valid with

$$\|f\|_1 \le \|f\|_{L^1(m_p)}, \quad f \in L^1(m_p). \tag{2.9}$$

Moreover, $L^1(m_p)$ is dense in $L^1(\mathbb{T})$.

Proof. (i) is clear from (1.7) and the definition of the Fourier transform. For, if $A \in \mathcal{B}(\mathbb{T})$ has Lebesgue measure zero, then $\widehat{\chi_B} = 0$ for all $B \in \Sigma$ with $B \subset A$. So, A is m_p-null. On the other hand, suppose that $A \in \mathcal{B}(\mathbb{T})$ is m_p-null. Then, in particular, $\widehat{\chi_A} = 0$ and hence, by injectivity of the Fourier transform, $\chi_A = 0$ in $L^p(\mathbb{T})$, that is, $|A| = 0$.

(ii) Fix $1 < p \le 2$ and an integer $N > 1$. Set $A_j = [2\pi(j-1)/N, 2\pi j/N)$ for $1 \le j \le N$. Then the sets A_j are pairwise disjoint and $|A_j| = 1/N$ for each $1 \le j \le N$. It is routine to check that

$$|\widehat{\chi_{A_1}}(m)| = 1/N, \quad 0 \le m < N,$$

and hence, that $\|\widehat{\chi_{A_1}}\|_{\ell^{p'}(\mathbb{Z})} \ge N^{-1/p}$. Since each function χ_{A_j} is a translate of χ_{A_1} for $1 < j \le N$, it follows that

$$\sum_{j=1}^{N} \|m_p(A_j)\|_{p'} = \sum_{j=1}^{N} \|\widehat{\chi_{A_j}}\|_{p'} \ge N^{1-\frac{1}{p}}.$$

Hence, m_p must have infinite variation.

(iii) We have, for $\chi_{\{0\}} \in (\ell^{p'}(\mathbb{Z}))' = \ell^p(\mathbb{Z})$, that

$$\langle m_p(A), \chi_{\{0\}} \rangle = \widehat{\chi_A}(0) = |A|$$

that is

$$|A| = |\langle m_p, \chi_{\{0\}} \rangle|(A), \quad A \in \mathcal{B}(\mathbb{T}).$$

According to (2.6), if $f \in L^1(m_p)$, then $\int_{\mathbb{T}} |f| \, dt = \int_{\mathbb{T}} |f| \, d|\langle m_p, \chi_{\{0\}} \rangle| < \infty$ and so $f \in L^1(\mathbb{T})$. Moreover, since $\|\chi_{\{0\}}\|_{\ell^p(\mathbb{Z})} = 1$, we obtain from (2.6) that

$$\|f\|_1 = \int_{\mathbb{T}} |f| \, dt = \int_{\mathbb{T}} |f| \, d|\langle m_p, \chi_{\{0\}} \rangle| \le \|f\|_{L^1(m_p)}.$$

This establishes $L^1(m_p) \subset L^1(\mathbb{T})$.

According to (i), the $\mathcal{B}(\mathbb{T})$-simple functions in $L^1(m_p)$ coincide with those in $L^1(\mathbb{T})$. Hence, $L^1(m_p)$ is dense in $L^1(\mathbb{T})$. □

Remark 2.2. For $p = 1$ the vector measure m_1 does have finite variation. Indeed, let $\{A_k\}$ be a Borel partition of $\mathbb{T}$. Then

$$\sum_k \|m_1(A_k)\|_\infty = \sum_k \|\widehat{\chi_{A_k}}\|_\infty \le \sum_k |A_k| = 1$$

and so $|m_1|(\mathbb{T})$ is finite.

A sublattice Z of $L^0(\mathbb{T})$ is an ideal if every $f \in L^0(\mathbb{T})$ satisfying $|f| \le |h|$ for some $h \in Z$ is necessarily itself in Z. If, in addition, there is a norm on Z such that Z is a Banach lattice relative to this norm, for the order induced from $L^0(\mathbb{T})$, then Z is called a Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$; see [Za, Ch.15]. Since the m_p-null sets and the Lebesgue null sets coincide, the previous recorded properties of the spaces $L^1(m)$, with m a general vector measure, when specialized to $L^1(m_p)$ imply that $L^1(m_p)$ is a Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$.

Lemma 2.3. *Let $1 \leq p \leq 2$. Then $L^p(\mathbb{T}) \subset L^1(m_p)$ and*

$$\int_A f dm_p = \widehat{f\chi_A}, \quad A \in \mathcal{B}(\mathbb{T}), \tag{2.10}$$

for every $f \in L^p(\mathbb{T})$. Moreover, we have

$$\|f\|_{L^1(m_p)} \leq 4\ \|f\|_p, \quad f \in L^p(\mathbb{T}). \tag{2.11}$$

Proof. To verify $L^p(\mathbb{T}) \subset L^1(m_p)$ it suffices to show that non-negative functions $f \in L^p(\mathbb{T})$ belong to $L^1(m_p)$. Choose simple functions $\{s_n\}_{n\in\mathbb{N}}$ with $0 \leq s_n \uparrow f$ pointwise on $\mathbb{T}$ and fix $A \in \mathcal{B}(\mathbb{T})$. Since $L^p(\mathbb{T})$ has σ-order continuous norm, we conclude that $\chi_A s_n \to \chi_A f$ in $L^p(\mathbb{T})$ as $n \to \infty$. By continuity of the Fourier transform map we obtain $\widehat{\chi_A s_n} \to \widehat{\chi_A f}$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. It is routine to check that

$$\int_B h\ dm_p = \widehat{\chi_B h}, \quad B \in \mathcal{B}(\mathbb{T}),$$

for every $\mathcal{B}(\mathbb{T})$-simple function h on $\mathbb{T}$. Hence, $\int_A s_n\ dm_p \to \widehat{\chi_A f}$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. According to [Le2, Theorem 2.4] the function $f \in L^1(m_p)$ and (2.10) holds.

To establish (2.11), let $f \in L^p(\mathbb{T})$. According to (2.7) and (2.10) we have

$$\|f\|_{L^1(m_p)} \leq 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|\widehat{\chi_A f}\|_{p'}.$$

By the Hausdorff-Young inequality

$$\|\widehat{\chi_A f}\|_{p'} \leq \|\chi_A f\|_p \leq \|f\|_p, \quad A \in \mathcal{B}(\mathbb{T}).$$

Hence, (2.11) holds. □

Corollary 2.4. *Let $1 \leq p \leq 2$. Then, for every $f \in L^1(m_p)$ and $A \in \mathcal{B}(\mathbb{T})$, we have*

$$\int_A f\ dm_p = \widehat{\chi_A f}. \tag{2.12}$$

In particular, the integration map I_{m_p} is a continuous extension of F from $L^p(\mathbb{T})$ to $L^1(m_p)$, still with values in $\ell^{p'}(\mathbb{Z})$.

Proof. Fix $f \in L^1(m_p)$. Choose simple functions $\{s_n\}_{n\in\mathbb{N}}$ with $s_n \to f$ in $L^1(m_p)$ as $n \to \infty$. By continuity of the integration map $I_{m_p} : L^1(m_p) \to \ell^{p'}(\mathbb{Z})$ and (2.10) we have, for $A \in \mathcal{B}(\mathbb{T})$,

$$\lim_{n\to\infty} \widehat{\chi_A s_n} = \lim_{n\to\infty} \int_A s_n\ dm_p = \int_A f\ dm_p,$$

with convergence in $\ell^{p'}(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$. On the other hand, (2.9) implies that $\chi_A s_n \to \chi_A f$ in $L^1(\mathbb{T})$ as $n \to \infty$ and so the continuity of $F : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ yields, for $A \in \mathcal{B}(\mathbb{T})$, that

$$\lim_{n\to\infty} \widehat{\chi_A s_n} = \widehat{\chi_A f},$$

with convergence in $\ell^\infty(\mathbb{Z})$. By uniqueness of Fourier transforms we see that (2.12) holds. □

Corollary 2.5. *Let $1 \le p \le 2$. Then the vector measure $m_p : \mathcal{B}(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ does not have relatively compact range.*

Proof. The closed convex hull of $m_p(\mathcal{B}(\mathbb{T}))$ is given by

$$C := \overline{co}\, m_p(\mathcal{B}(\mathbb{T})) = \{\int_{\mathbb{T}} f \; dm_p \,:\, 0 \le f \le 1,\; f \in L^\infty(m_p)\},$$

[DU, p.263]. Moreover, according to (2.12) each character e_n, for $n \in \mathbb{Z}$, satisfies

$$\chi_{\{n\}} = F(e_n) = \int_{\mathbb{T}} e_n \; dm_p \in C + C + iC + iC.$$

So, if $m_p(\mathcal{B})$ is relatively compact in $\ell^{p'}(\mathbb{Z})$, then so is $C + C + iC + iC$ and hence, also $\{\chi_{\{n\}} \,:\, n \in \mathbb{Z}\}$. But, this is surely not the case as $\|\chi_{\{n\}} - \chi_{\{k\}}\|_{p'} = 2^{1/p'}$ for $n \ne k$. □

Proof of Theorem 1.1 We show that $\mathbf{F}^p(\mathbb{T}) := L^1(m_p)$, equipped with the norm $\|\cdot\|_{\mathbf{F}^p(\mathbb{T})} := \|\cdot\|_{L^1(m_p)}$, has all the required features. As already noted, $L^1(m_p)$ is a Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$ with σ-order continuous norm. Part (i) of Theorem 1.1 is immediate from Lemma 2.3 and Corollary 2.4 and part (iii) is clear from Lemma 2.1 (iii).

To establish (ii), let Z be any Banach function space over $(\mathbb{T}, \mathcal{B}(\mathbb{T}), dt)$ with σ-order continuous norm such that $L^p(\mathbb{T}) \subset Z$ continuously and F has a continuous linear extension $T : Z \to \ell^{p'}(\mathbb{Z})$. Let $0 \le f \in Z$. Choose simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that $0 \le s_n \uparrow f$ pointwise a.e. on $\mathbb{T}$ and note that $\{s_n\}_{n\in\mathbb{N}} \subset L^p(\mathbb{T}) \subset Z$. Fix $A \in \mathcal{B}(\mathbb{T})$. Since Z has σ-order continuous norm, it follows that $T(s_n\chi_A) \to T(f\chi_A)$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. But, for $n \in \mathbb{N}$ we have

$$T(s_n\chi_A) = F(s_n\chi_A) = \int_A s_n \; dm_p, \tag{2.13}$$

and so $\int_A s_n \; dm_p \to T(f\chi_A)$ in $\ell^{p'}(\mathbb{Z})$ as $n \to \infty$. Again by [Le2, Theorem 2.4] it follows that $f \in L^1(m_p)$ and

$$\int_A f \; dm_p = \lim_{n\to\infty} \int_A s_n \; dm_p = T(f\chi_A). \tag{2.14}$$

The case for general $f \in Z$ follows by considering the positive and negative parts of both $Re(f)$ and $Im(f)$, all of which belong to Z. So, $Z \subset L^1(m_p)$. It remains to verify the continuity of this inclusion. Given $f \in Z$, it follows from (2.7) and (2.14) that

$$\|f\|_{L^1(m_p)} \le 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|\int_A f \; dm_p\|_{p'} = 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|T(\chi_A f)\|_{p'}.$$

By continuity, $\|T(\chi_A f)\|_{p'} \le \|T\| \, \|f\chi_A\|_Z$ and, since the norm on Z is a lattice norm and $|f\chi_A| \le |f|$, also $\|f\chi_A\|_Z \le \|f\|_Z$ for each $A \in \mathcal{B}(\mathbb{T})$. It follows that

$$\|f\|_{L^1(m_p)} \le 4\, \|T\| \, \|f\|_Z.$$

This completes the proof of Theorem 1.1.

Remark 2.6. We can now justify the claims made in Remark 1.3. Since $L^1(m_1) = \mathbf{F}^1(\mathbb{T})$ and also $L^1(m_1) = L^1(\mathbb{T})$, by Lemma 2.1 (iii) and Lemma 2.3 (with equivalent norms), it follows that $F^1(\mathbb{T}) = L^1(\mathbb{T})$.

For $p = 2$, the Plancherel theorem and (2.11) yield, for each $f \in L^2(\mathbb{T})$, that

$$\|f\|_{L^1(m_2)} \leq 4\ \|f\|_2 = 4\ \|\hat{f}\|_2 \leq 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|\widehat{\chi_A f}\|_2.$$

Then apply (2.7) and (2.12) to conclude, for $f \in L^2(\mathbb{T})$, that

$$\|f\|_{L^1(m_2)} \leq 4\ \|f\|_2 \leq 4\ \|f\|_{L^1(m_2)}.$$

Moreover, by Lemma 2.3, $L^2(\mathbb{T})$ is contained and dense in $L^1(m_2)$. It follows that $L^1(m_2) = L^2(\mathbb{T})$.

3 Proof of Theorem 1.2

To describe the space $L^1(m)$, for a general vector measure m, is rather difficult. However, for the vector measures m_p, with $1 \leq p \leq 2$, it will be shown in this section that this is possible.

According to (1.2), $V^p(\mathbb{T}) = \{h \in L^{p'}(\mathbb{T}) : h = \check{\phi} \text{ for some } \phi \in \ell^p(\mathbb{Z})\}$. Since $1 \leq p \leq 2$, we have $\ell^p(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and so the inverse Fourier transform $\check{\phi} \in L^2(\mathbb{T})$ for $\phi \in \ell^p(\mathbb{Z})$. For $1 \leq p < 2$ the containment

$$V^p(\mathbb{T}) \subset L^{p'}(\mathbb{T}) \tag{3.1}$$

is always proper; see [Ka, p.101] for $1 < p < 2$. For $p = 1$, note that $V^1(\mathbb{T}) = \{\hat{\phi} : \phi \in \ell^1(\mathbb{Z})\} \subset C(\mathbb{T})$ with a proper containment, [Ka, p.31].

Lemma 3.1. *Let $1 \leq p \leq 2$ and $f \in L^0(\mathbb{T})$. Then $f \in L^1(m_p)$ if and only if*

$$\int_{\mathbb{T}} |f|\ |h|\ dt < \infty, \quad h \in V^p(\mathbb{T}). \tag{3.2}$$

Proof. Suppose $f \in L^1(m_p)$. If $h = \check{\phi} \in V^p(\mathbb{T})$ for some $\phi \in \ell^p(\mathbb{Z})$, then (2.3) implies that $\int_{\mathbb{T}} |f|\ d|\langle m_p, \phi\rangle| < \infty$. Since $L^{p'}(\mathbb{T}) \subset L^2(\mathbb{T})$, we can apply Parseval's formula to conclude, for each $A \in \mathcal{B}(\mathbb{T})$, that

$$\langle m_p(A), \phi\rangle = \langle \widehat{\chi_A}, \hat{h}\rangle = \langle \chi_A, \tilde{h}\rangle = \int_A \tilde{h}\ dt,$$

where $\tilde{h}(t) = h(-t)$ is the reflection of h. Accordingly, the variation measure $|\langle m_p, \phi\rangle|(A) = \int_A |\tilde{h}|\ dt$ for $A \in \mathcal{B}(\mathbb{T})$ and therefore

$$\int_{\mathbb{T}} |f|\,|\tilde{h}|\ dt = \int_{\mathbb{T}} |f|\ d|\langle m_p, \phi\rangle| < \infty. \tag{3.3}$$

Since $V^p(\mathbb{T})$ is invariant under formation of reflections, (3.2) holds. Conversely, let $1 < p \le 2$ and suppose that $f \in L^0(\mathbb{T})$ satisfies (3.2). Given any $\phi \in \ell^p(\mathbb{Z})$ there exists $h \in L^{p'}(\mathbb{T})$ such that $\hat{h} = \phi$, [Ka, IV Theorem 2.2]. Then $h \in V^p(\mathbb{T})$ and hence, also $\tilde{h} \in V^p(\mathbb{T})$. So, $\int_{\mathbb{T}} |f|\,|\tilde{h}|\,dt < \infty$. Moreover, the same calculation as above shows that the equality in (3.3) holds and hence, is finite. So, f satisfies (2.3). Since the reflexive space $\ell^{p'}(\mathbb{Z})$ cannot contain an isomorphic copy of the Banach space c_0, this alone suffices to ensure that $f \in L^1(m_p)$, [Le, Theorem 5.1]. For $p = 1$, note that the constant function $1 = \check{\chi}_{\{0\}}$ belongs to $V^1(\mathbb{T})$ and so $\int_{\mathbb{T}} |f|\,dt = \int_{\mathbb{T}} |f|\,\check{\chi}_{\{0\}}\,dt < \infty$, that is, $f \in L^1(\mathbb{T}) = L^1(m_1)$; see Remark 2.6. □

Fix $1 < p \le 2$ and let $f \in \Phi^p(\mathbb{T})$; see (1.5). That is, $f \in L^1(\mathbb{T})$ has the property that $\widehat{\chi_A f} \in \ell^{p'}(\mathbb{Z})$ for every $A \in \mathcal{B}(\mathbb{T})$. Then the set function $\nu_f : \mathcal{B}(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ defined by

$$A \mapsto \nu_f(A) := \widehat{\chi_A f}, \quad A \in \mathcal{B}(\mathbb{T}), \tag{3.4}$$

is surely finitely additive. Actually more is true.

Lemma 3.2. *Let $1 < p \le 2$. Then, for each $f \in \Phi^p(\mathbb{T})$, the finitely additive set function ν_f as given by (3.4) is σ-additive, that is, it is an $\ell^{p'}(\mathbb{Z})$-valued vector measure on $\mathcal{B}(\mathbb{T})$.*

Proof. Let Γ denote the linear span of $\chi_{\{m\}}$, for $m \in \mathbb{Z}$, and let $\{A_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of sets in $\mathcal{B}(\mathbb{T})$. Let $\{A_{n_k}\}_{k\in\mathbb{N}}$ be any subsequence of $\{A_n\}_{n\in\mathbb{N}}$. Then, with $B = \cup_{k\in\mathbb{N}} A_{n_k}$, the dominated convergence theorem gives, for each $m \in \mathbb{Z}$, that

$$\sum_{k\in\mathbb{N}} \langle \nu_f(A_{n_k}), \chi_{\{m\}}\rangle = \sum_{k\in\mathbb{N}} \int_{\mathbb{T}} f(t)\,\chi_{A_{n_k}}(t)\,e^{-imt}\,dt = \widehat{f\chi_B}(m) = \sum_{k\in\mathbb{N}} \langle \nu_f(\cup_{k\in\mathbb{N}} A_{n_k}), \chi_{\{m\}}\rangle. \tag{3.5}$$

So, every subseries of $\sum_{k\in\mathbb{N}} \nu_f(A_n)$ is weakly Γ-convergent. Since the reflexive space $\ell^{p'}(\mathbb{Z})$ cannot contain an isomorphic copy of ℓ^∞ and Γ is a total subset of $(\ell^{p'}(\mathbb{Z}))' = \ell^p(\mathbb{Z})$, it follows from the strengthened version of the Orlicz-Pettis theorem, [DU, p.23], that $\nu_f(\cup_{n\in\mathbb{N}} A_n)$ is unconditionally norm convergent (to $\nu_f(B)$). Accordingly, ν_f is σ-additive. □

Proposition 3.3. *Let $1 < p \le 2$. Then $L^1(m_p) = \Phi^p(\mathbb{T})$.*

Proof. By Corollary 2.4 and (1.5) it is clear that $L^1(m_p) \subset \Phi^p(\mathbb{T})$. Conversely, suppose that $f \in \Phi^p(\mathbb{T})$. Given $h \in V^p(\mathbb{T})$, there is $\phi \in \ell^p(\mathbb{Z})$ such that $\hat{h} = \phi$ and

$$\langle m_p, \phi\rangle(A) = \int_A \tilde{h}\,dt, \quad A \in \mathcal{B}(\mathbb{T}); \tag{3.6}$$

see the proof of Lemma 3.1. According to Lemma 3.2,

$$A \mapsto \langle \nu_f(A), \phi\rangle = \langle \widehat{f\chi_A}, \hat{h}\rangle, \quad A \in \mathcal{B}(\mathbb{T}),$$

is σ-additive. Define $A_n = |f|^{-1}([0,n])$ for $n \in \mathbb{N}$, in which case $A \cap A_n \uparrow A$ for each $A \in \mathcal{B}(\mathbb{T})$. By σ-additivity of $\langle \nu_f(A), \phi \rangle$ we have

$$\langle \nu_f(A), \phi \rangle = \lim_{n\to\infty} \langle \widehat{f\chi_{A\cap A_n}}, \hat{h} \rangle.$$

Since each function $f\chi_{A\cap A_n}$ is bounded and $h \in L^2(\mathbb{T})$, Parseval's formula gives

$$\langle \nu_f(A), \phi \rangle = \lim_{n\to\infty} \int_{\mathbb{T}} f\chi_{A\cap A_n} \tilde{h}\ dt = \lim_{n\to\infty} \int_A f_n d\mu,$$

where the functions $f_n = f\chi_{A_n} \in L^\infty(\mathbb{T})$ converge pointwise to f on $\mathbb{T}$ and $d\mu = \tilde{h}\, dt$ is a complex measure. By [Le2, Lemma 2.3] we conclude that f is μ-integrable (i.e. $f\tilde{h} \in L^1(\mathbb{T})$) and

$$\int_A f\tilde{h}\, dt = \int_A f d\mu = \langle \nu_f(A), \phi \rangle.$$

So, $f\tilde{h} \in L^1(\mathbb{T})$ for all $h \in V^p(\mathbb{T})$. Then Lemma 3.1 implies that $f \in L^1(m_p)$. □

We have an immediate consequence for the spaces $\Gamma^p(\mathbb{T})$ as given by (1.6).

Corollary 3.4. *For each* $1 < p \le 2$ *we have* $L^1(m_p) = \Gamma^p(\mathbb{T})$.

Proof. Let $f \in \Gamma^p(\mathbb{T})$. Then the operator S_f (see (1.3)) maps each $h \in L^\infty(\mathbb{T})$ into $\ell^{p'}(\mathbb{Z})$. In particular, for $h = \chi_A$ we have

$$S_f(\chi_A) = \widehat{f\chi_A} \in \ell^{p'}(\mathbb{Z}), \quad A \in \mathcal{B}(\mathbb{T}),$$

that is, $f \in \Phi^p(\mathbb{T})$. By Proposition 3.3 we have $f \in L^1(m_p)$.

Conversely, suppose that $f \in L^1(m_p)$. Given $h \in L^\infty(\mathbb{T})$, we have a.e. $|h| \le \|h\|_\infty\ \chi_{\mathbb{T}}$. Since the Lebesgue null sets and m_p-null sets coincide, we also have

$$|h| \le \|h\|_\infty\ \chi_{\mathbb{T}}, \quad m_p - \text{a.e.} \tag{3.7}$$

In particular, $h \in L^\infty(m_p)$ and so $hf \in L^1(m_p)$ by the ideal property of the Banach function space $L^1(m_p)$. Then Corollary 2.4 can be applied to yield $S_f(h) = \widehat{fh} = \int_{\mathbb{T}} fh\ dm_p \in \ell^{p'}(\mathbb{Z})$ and hence, that

$$\|S_f(h)\|_{p'} = \| \int_{\mathbb{T}} fh\ dm_p\|_{p'} \le \|fh\|_{L^1(m_p)}.$$

Since the norm of $L^1(m_p)$ is a lattice norm, by (3.7) we get $\|fh\|_{L^1(m_p)} \le \|h\|_\infty\ \|f\|_{L^1(\mathbb{T})}$. Accordingly,

$$\|S_f(h)\|_{p'} \le \|h\|_\infty\ \|f\|_{L^1(m_p)}.$$

This shows that S_f is a bounded operator from $L^\infty(\mathbb{T})$ to $\ell^{p'}(\mathbb{Z})$ with $\|S_f\|_{\infty,p'} \le \|f\|_{L^1(m_p)}$. In particular, $f \in \Gamma^p(\mathbb{T})$. □

Proof of Theorem 1.2. Since $\mathbf{F}^p(\mathbb{T}) = L^1(m_p)$, it follows from Lemma 3.1 that $\mathbf{F}^p(\mathbb{T}) = \Delta^p(\mathbb{T})$, for all $1 \leq p \leq 2$. It is clear from (1.5) and (1.6) that $\Phi^1(\mathbb{T}) = \Gamma^1(\mathbb{T}) = L^1(\mathbb{T})$ and hence, $\Phi^1(\mathbb{T}) = \Gamma^1(\mathbb{T}) = \mathbf{F}^1(\mathbb{T})$ by Remark 2.6. For $1 < p \leq 2$, it follows from Proposition 3.3 that $\mathbf{F}^p(\mathbb{T}) = \Phi^p(\mathbb{T})$ and from Corollary 3.4 that $\mathbf{F}^p(\mathbb{T}) = \Gamma^p(\mathbb{T})$. Moreover, by (2.7) and Corollary 2.4 we have

$$\|f\|_{L^1(m_p)} \leq 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|\int_{\mathbb{T}} \chi_A f\ dm_p\|_{p'} = 4 \sup_{A\in\mathcal{B}(\mathbb{T})} \|S_f(\chi_A)\|_{p'} \leq 4\ \|S_f\|_{\infty,p'}.$$

Accordingly, for $f \in L^1(m_p) = \Gamma^p(\mathbb{T})$ the norms $\|f\|_{L^1(m_p)}$ and $\|S_f\|_{\infty,p'}$ are equivalent. $\square$

In the remainder of this section we consider some properties of the optimal lattice domain $\mathbf{F}^p(\mathbb{T}) = L^1(m_p)$.

Fix $1 < p < 2$. The *associate space* of the Banach function space $\mathbf{F}^p(\mathbb{T})$ consists of all $h \in L^0(\mathbb{T})$ satisfying

$$\int_{\mathbb{T}} |fh|\ dt < \infty, \qquad f \in \mathbf{F}^p(\mathbb{T}), \tag{3.8}$$

equipped with the norm $\sup\{\int_{\mathbb{T}} |fh|\ dt\ :\ \|f\|_{\mathbf{F}^p(\mathbb{T})} = 1\}$, [Za, Ch.15, Sect. 69]. Since $\mathbf{F}^p(\mathbb{T})$ has σ-order continuous norm, the Banach space dual of $\mathbf{F}^p(\mathbb{T})$ coincides with its associate space, [Za, p.480]. Moreover, $(\mathbf{F}^p(\mathbb{T}))'$ is again a Banach function space in $L^0(\mathbb{T})$, [Za, p.457]. In particular, it is an ideal in $L^0(\mathbb{T})$. As noted in the proof of Lemma 3.1, a function $f \in L^0(\mathbb{T})$ belongs to $L^1(m_p)$ if and only if it satisfies (2.3). This implies that $L^1(m_p) = \mathbf{F}^p(\mathbb{T})$ is *weakly sequentially complete*, has the *σ-Fatou property* (i.e. $0 \leq f_n \uparrow f$ with $\{f_n\} \subset \mathbf{F}^p(\mathbb{T})$ a norm bounded sequence implies that $\|f_n\|_{\mathbf{F}^p(\mathbb{T})} \uparrow \|f\|_{\mathbf{F}^p(\mathbb{T})}$) and that $L^1(m_p)$ coincides with its second associate space, [CR4, Prop. 2.1, 2.3, 2.4].

Note that the operator norm of $S_f \in \mathcal{L}(L^\infty(\mathbb{T}), \ell^{p'}(\mathbb{Z}))$, for $f \in \mathbf{F}^p(\mathbb{T})$, agrees with the norm of the dual operator

$$S_f^* : \ell^p(\mathbb{Z}) \ni \{a_n\} \mapsto f(x) \sum_{n\in\mathbb{Z}} a_n e^{-inx} \in L^1(\mathbb{T}) \subset L^\infty(\mathbb{T})'.$$

Since $\ell^p(\mathbb{Z})$ is modulation invariant, i.e. $\{a_n\}$ and $\{e^{in\alpha}a_n\}$ have the same norm, it is clear that $\mathbf{F}^p(\mathbb{T})$ is translation invariant. Moreover, it is easy to check that the translation operators $\tau_t f(x) = f(x-t)$ are continuous in $\mathbf{F}^p(\mathbb{T})$ and that τ_t converges to the identity for the strong operator topology as $t \to 0$. Accordingly, $\mathbf{F}^p(\mathbb{T})$ is a *homogeneous Banach space*, [Ka].

If $h \in V^p(\mathbb{T})$, then (3.8) holds because of (1.4) and Theorem 1.2. From the natural inclusion $L^p(\mathbb{T}) \subset \mathbf{F}^p(\mathbb{T})$ we then conclude that

$$V^p(\mathbb{T}) \subset (\mathbf{F}^p(\mathbb{T}))' \subset L^{p'}(\mathbb{T}). \tag{3.9}$$

Actually, since $\chi_{\mathbb{T}} \in V^p(\mathbb{T})$ and $(\mathbf{F}^p(\mathbb{T}))'$ is an ideal, we see that also $L^\infty(\mathbb{T}) \subset (\mathbf{F}^p(\mathbb{T}))'$. It follows easily from (3.8) that $(\mathbf{F}^p(\mathbb{T}))'$ is translation invariant. According to [Ka, IV Theorem 2.4], there exists $h \in C(\mathbb{T})$ such that $\hat{h} \notin \ell^p(\mathbb{Z})$. If the first containment in (3.9) was an equality, then $V^p(\mathbb{T})$ would be an ideal and so the inequality $|h| \le \|h\|_\infty \chi_\mathbb{T}$ would give that $h \in V^p(\mathbb{T})$, which is not the case. Of course, since $V^p(\mathbb{T})$ contains the trigonometric polynomials, it surely separates the points of $\mathbf{F}^p(\mathbb{T})$. If the second containment in (3.9) was an equality, then $L^p(\mathbb{T})$ would coincide with the second associate space of $L^1(m_p)$ which, as noted above, equals $L^1(m_p) = \mathbf{F}^p(\mathbb{T})$. This contradicts Theorem 1.4. So, both containments in (3.9) are proper.

From the viewpoint of analysis, the weak sequential completeness of $\mathbf{F}^p(\mathbb{T})$ is difficult to use in practice since $(\mathbf{F}^p(\mathbb{T}))'$ is not explicitly known. However, there is available a good substitute in this regard. Indeed, Theorem 1.1 (iii) and the σ-Fatou property of $\mathbf{F}^p(\mathbb{T})$ show that $\mathbf{F}^p(\mathbb{T})$ is also a Banach function space in the more restricted sense of [BS]. Since $L^\infty(\mathbb{T})$ is an order ideal of $(\mathbf{F}^p(\mathbb{T})))'$ containing the simple functions , it follows from [BS, Ch.1, Theorem 5.2] that $\mathbf{F}^p(\mathbb{T})$ is also sequentially $\sigma(\mathbf{F}^p(\mathbb{T}), L^\infty(\mathbb{T}))$-complete.

4 Proof of Theorem 1.4

The proof of Theorem 1.4, for $p' > 2$ an even integer, is somewhat easier because in this case we can rely on the Hardy-Littlewood majorant property of the spaces $L^p(\mathbb{T})$, [HL]. To see this and to get an idea of what type of functions are contained in $\mathbf{F}^p(\mathbb{T})$ we establish, e.g. for $p = 4/3$, the following

Lemma 4.1. *If $f \in L^1(\mathbb{T})$ is non-negative and $\hat{f} \in \ell^4(\mathbb{Z})$, then $f \in \mathbf{F}^{4/3}(\mathbb{T})$ and*

$$\|f\|_{\mathbf{F}^{4/3}(\mathbb{T})} \le 4\,\|\hat{f}\|_{\ell^4(\mathbb{Z})}.$$

Proof. This follows from Parsevals's identity as follows. For $g \in L^\infty(\mathbb{T})$ we have

$$\|\widehat{fg}\|_4^4 = \sum_{n\in\mathbb{Z}} |\widehat{fg}(n)\ \widehat{fg}(n)|^2 = \int_\mathbb{T} |fg * fg|^2\,dt \le \|g\|_\infty^4 \int_\mathbb{T} |f * f|^2\,dt = \|g\|_\infty^4\ \|\hat{f}\|_4^4,$$

that is, $\|S_f\|_{\infty,4} \le \|\hat{f}\|_4$. Hence, $\|f\|_{\mathbf{F}^{4/3}(\mathbb{T})} \le 4\,\|S_f\|_{\infty,4} \le 4\,\|\hat{f}\|_4$. □

What we have said so far applies also for higher dimensional tori $\mathbb{T}^d \cong (-\pi,\pi]^d$. In particular, we may apply the previous Lemma to $\mathbb{T}^2$ to see that $L^{4/3}(\mathbb{T}^2)$ is a proper subspace of $\mathbf{F}^{4/3}(\mathbb{T}^2)$. In fact, for $\alpha > 0$, the Fourier transform of the non-negative function $M_\alpha : x \mapsto \frac{1}{\Gamma(\alpha)}(1-|x|^2)_+^{\alpha-1}$, defined on $(-\pi,\pi]^2$, decays asymptotically as $|n|^{-\frac{1}{2}-\alpha}$ for $n \to \infty$ in $\mathbb{Z}^2$. Therefore, $M_\alpha \in \mathbf{F}^{4/3}(\mathbb{T}^2)$ for all $\alpha > 0$, whereas M_α is obviously not an $L^{4/3}(\mathbb{T}^2)$-function for $\alpha \le 1/4$.

We note that for $\alpha \to 0$ the functions M_α, considered as distributions on $\mathbb{T}^2$, converge to arclength measure $d\sigma$ on the circle S^1. Hence, for $\alpha \to 0$, the L^1-function M_α does not converge in the $\mathbf{F}^{4/3}(\mathbb{T}^2)$-norm. However, it was shown by E.M. Stein (see [Fe], [St]) that the operator

$$S_\sigma(f) = \widehat{fd\sigma}, \quad f \in C^\infty(S^1),$$

maps $L^2(S^1, d\sigma)$ and hence, also $L^\infty(S^1, d\sigma)$, boundedly into $L^q(\mathbb{R}^2)$ for some $q > 4$. The fact that S_σ maps $L^\infty(S^1, d\sigma)$ into $L^q(\mathbb{R}^2)$ for all $q > 4$ was also shown in [Fe]; see also [St]. An easy argument shows that we may replace $L^q(\mathbb{R}^2)$ with $\ell^q(\mathbb{Z}^2)$.

This motivates us to use Fourier restriction theory to establish that the inclusion $L^p(\mathbb{T}) \subset \mathbf{F}^p(\mathbb{T})$ is proper. For the proof of Theorem 1.4 we will employ the following result for Salem measures, [Mo, Mo2, Sa].

Proposition 4.2. *There is a non-negative measure μ on $\mathbb{R}$ with the following properties.*

(i) *$E = \operatorname{supp}(\mu)$ is a compact subset of $[-1, 1] \subset (-\pi, \pi]$ of Hausdorff dimension $\alpha \in (0, 1)$.*

(ii) *There is $C > 0$ such that, for each interval $I \subset \mathbb{R}$, we have*

$$\mu(I) \le C\ |I|^\alpha.$$

(iii) *For each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that the Fourier transform of μ (on $\mathbb{R}$) satisfies the asymptotic bound*

$$|\widehat{\mu}(\xi)| \le C_\varepsilon\ |\xi|^{-\frac{\alpha}{2}+\varepsilon}, \quad |\xi| \to \infty.$$

(iv) *The following analogue of the Stein-Tomas restriction inequality holds:*

$$\int |\hat{f}(y)|^2\ d\mu(y) \le C\ \|f\|^2_{L^p(\mathbb{R})}, \qquad f \in L^p(\mathbb{R}), \tag{4.1}$$

for $1 \le p < p_\varepsilon(\alpha)$, where $p_\varepsilon(\alpha) \to \frac{2(2-\alpha)}{4-3\alpha}$ as $\varepsilon \to 0^+$. □

We will need to transfer inequality (4.1) to the torus. For a sequence $\{f_m\} \in \ell^p(\mathbb{Z})$ we consider $f(x) = \sum_{m\in\mathbb{Z}} f_m\ \chi_Q(2\pi m + x)$, where $Q = (-\pi, \pi]$ is a fundamental interval for the lattice $2\pi\mathbb{Z}$. Obviously we have $f \in L^p(\mathbb{R})$. Now apply (4.1) to f. Since $\hat{f}(\xi) = \widehat{\chi_Q}(\xi) \sum_{m\in\mathbb{Z}} f_m e^{im\xi} =: \widehat{\chi_Q}(\xi) F(\xi)$ and $|\chi_Q(\xi)| > 1/2$ on E, we obtain for the periodic function F the inequality

$$\int |F|^2\ d\mu \le C\ \big(\sum_{m\in\mathbb{Z}} |f_m|^p\big)^{2/p}. \tag{4.2}$$

Hence, for each $g \in L^\infty(E, d\mu)$, we get by the dual inequality of (4.2) that

$$\Big(\sum_{m\in\mathbb{Z}} |\widehat{g d\mu}(m)|^{p'}\Big)^{\frac{1}{p'}} \le C\ \|g\|_{L^2(E,d\mu)} \le C\ \|g\|_{L^\infty(E,d\mu)}.$$

Denote by μ_t the translation of μ by $t \in \mathbb{R}$, let I be an open interval centred at 0 of length $1/10$, let $\phi \in C^\infty(I)$ be non-negative with $\phi(0) = 1 = \widehat{\phi}(0)$ and, for $0 < \beta < 1$, define $r_\beta(t) = |t|^{-\beta}\phi(t)$. Now define the non-negative function

$$I_\beta(y) = \int_{\mathbb{R}} r_\beta(t)\ d\mu_t(y) = (r_\beta * \mu)(y), \quad y \in \mathbb{R}.$$

Note that $I_\beta \in L^1(\mathbb{R})$ and $\operatorname{supp} I_\beta$ is a proper subset of Q. Clearly, the left-hand side of (4.1) is translation invariant and so

$$\int |\hat{f}(y)|^2 \, d\mu_t(y) \le C \, \|f\|^2_{L^p(\mathbb{R})}, \qquad t \in \mathbb{R}.$$

Multiplying by r_β and then integrating with respect to t, gives

$$\int |\hat{f}(y)|^2 \, I_\beta(y) \, dy \le C \, \|f\|^2_{L^p(\mathbb{R})}.$$

As above we obtain, for $F(x) = \sum_{m\in\mathbb{Z}} f_m \, e^{imx}$, that

$$\int_Q |F|^2 \, I_\beta(y) \, dy \le C \, \Big(\sum_{m\in\mathbb{Z}} |f_m|^p \Big)^{2/p} \tag{4.3}$$

and therefore, for each $g \in L^\infty(\mathbb{T})$, that

$$\Big(\sum_{m\in\mathbb{Z}} |\widehat{g\, I_\beta}(m)|^{p'} \Big)^{\frac{1}{p'}} \le C \, \|g\|_{L^2(Q, I_\beta(y)dy)} \le C \, \|g\|_{L^\infty(Q)} \int_Q I_\beta(y) dy. \tag{4.4}$$

That is, $I_\beta \in \mathbf{F}^p(\mathbb{T})$ for $0 < \beta < 1$ and $1 \le p < p_\varepsilon(\alpha)$.

Proof of Theorem 1.4. We will show, for an appropriate choice of α and β, that $I_\beta \notin L^p(Q)$. Note that:

- from Proposition 4.2 (ii) we obtain $I_\beta \in L^\infty(Q)$ for $\beta < \alpha$ and, of course, $I_\beta \in L^1(Q)$ for $0 < \beta < 1$. Therefore, by convexity we get, for $\beta > \alpha$, that

$$I_\beta \in L^p(Q), \quad \text{ if } p < \frac{1-\alpha}{\beta-\alpha}. \tag{4.5}$$

  Below we will see that this condition is essentially sharp.

- Since $|\widehat{r_\beta}(\xi)| \ge c \, |\xi|^{-(1-\beta)}$ as $|\xi| \to \infty$, we obtain

$$|\widehat{I_\beta}(\xi)| = |\widehat{\mu}(\xi) \, \widehat{r_\beta}(\xi)| \ge c \, |\widehat{\mu}(\xi)| \, |\xi|^{-(1-\beta)}, \quad |\xi| \to \infty.$$

  Therefore

$$\int |\widehat{I_\beta}(\xi)|^2 \, d\xi \ge c \int |\widehat{\mu}(\xi)|^2 \, (1+|\xi|)^{-2(1-\beta)} \, d\xi \approx I_t(d\mu),$$

  where $t = 2\beta - 1$ and $I_t(d\mu)$ is the t-energy of μ (see [Fa]). From property (iii) in Proposition 4.2 we obtain that $t \le \dim_H E = \alpha$ (see [Fa, p.79]). That is, $\beta \le \frac{1+\alpha}{2}$ provided $\int_{\mathbb{R}} |\widehat{I_\beta}(\xi)|^2 \, d\xi$ is finite.

Suppose now that $\beta_0 > \alpha$ and $I_{\beta_0} \in L^{q_0}(Q)$ for $q_0 := \frac{1-\alpha+\delta}{\beta_0-\alpha}$ and some $\delta > 0$. Since $I_\beta \in L^\infty(Q)$ for $\beta < \alpha$, by convexity we obtain that $I_\beta \in L^2(\mathbb{R})$ for all $\beta < \frac{1+\alpha}{2} + \frac{\delta}{2}$. Hence, $\widehat{I_\beta} \in L^2(\mathbb{R})$, that is, the t-energy of μ is finite for all $t = 2\beta - 1 < \alpha + \delta$. Accordingly, $\delta = 0$.

Hence, for a given $p \in (1,2)$ we may choose $\alpha \in (0,1)$ such that $\frac{2(2-\alpha)}{4-3\alpha} > p$. By choosing $\beta > \alpha$ sufficiently close to 1 we can ensure that $I_\beta \notin L^p(\mathbb{T})$, but $I_\beta \in \mathbf{F}^p(\mathbb{T})$. $\square$

We conclude with the observation that $L^r(\mathbb{T}) \nsubseteq \mathbf{F}^p(\mathbb{T})$ for $1 \le r < p$ and $\mathbf{F}^p(\mathbb{T}) \nsubseteq L^r(\mathbb{T})$ for $1 < r \le p$. The first statement follows by considering $f(t) = |t|^{-1/p}$. On the other hand, the above construction ensures, for any $r \in (1,p)$, that the space $\mathbf{F}^p(\mathbb{T})$ is not contained in $L^r(\mathbb{T})$. This is not surprising, since L^p-spaces merely measure a local property, whereas the $\mathbf{F}^p(\mathbb{T})$-norm involves not only local properties but also arithmetic properties of a function (e.g.in case of I_β, "by lack of a better description" this means, not only are its peaks important but also how they are distributed relative to each other). One may also see this by estimating the $\mathbf{F}^p(\mathbb{T})$-norm of $f(mx)$ for $m \in \mathbb{N}$ and $f \in \mathbf{F}^p(\mathbb{T})$.

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