On the Hardy-Littlewood majorant problem for random sets

G. Mockenhaupt & W. Schlag

1 The majorant property: Introduction

This paper is concerned with versions of the majorant property of various randomly generated subsets of integers in $[1,N]$. More precisely, suppose $A\subset[1,N]$ is a set of integers of size $|A|\asymp N^\rho$ for some fixed $0<\rho<1$. For example, one can take A to be the squares, cubes, etc., or (multi-dimensional) arithmetic progressions in $[1,N]$. Given $p\geq 2$ we ask for the smallest constant C such that uniformly for $|a_n|$ majorized by 1 (we write $e(nt)=e^{2\pi int}$)

$$\Big\|\sum_{n\in A} a_n\, e(n\cdot)\Big\|_p \leq\, C\,\Big\|\sum_{n\in A} e(n\cdot)\Big\|_p. \tag{1.1}$$

If p is an even integer, then one can take $C=1$, in particular C does not depend on A (resp. N). In fact, Hardy and Littlewood [HL] realized that whenever $|a_n|\leq b_n$ and p even

$$\Big\|\sum_{n=1}^{N} a_n\, e(n\cdot)\Big\|_p \leq\, C\,\Big\|\sum_{n=1}^{N} b_n\, e(n\cdot)\Big\|_p \tag{1.2}$$

holds with $C=1$. On the other hand, it has been known for some time that if p is not an even integer the constant C in (1.2) does grow unboundedly with N (see, e.g. [Mont, p.133]). A quantitative lower bound of order $N^{c/\log\log N}$, for some $c>0$, is obtained in [M] for (1.1) with a particular sequence of integer sets A_N in $[1,N]$. We will improve on this lower bound and show that for an appropriate sequence of integer sets $A_N\subset[1,N]$ the constants in (1.1) grow by a power in N (see Theorem (3.2)). This result has also been obtained with a similar method by B. Green and I. Ruzsa [GR] for the case $p=3$. Unfortunately, both methods do not reveal a structural property of sets A which would guaranty a power growth in N of the constant C in (1.1).

Inequality (1.1) for particular sets A plays an important role in analysis and number theory. For example, it is conjectured by H. Montgomery (see [Mont, p.11]) that for $2<p<4$ the frequency sets

$$A=\{\,[N^{p/2}\log n]\mid 1\leq n\leq N\},$$

here $[\cdot]$ denotes the integer part, satisfy (1.1) with a slow growing bound $C=C_\varepsilon N^\varepsilon, \varepsilon>0$. We may also interpret (1.1) for certain sets A as a reformulations of the restriction conjecture for the Fourier transform on $\mathbb{R}^d$:

$$\|\widehat{f}\|_{L^1(S^{d-1},d\sigma)}\,\leq\, C\,\|f\|_{L^p(\mathbb{R}^d)},\quad p<2d/(d+1),$$

here σ is rotational invariant measure on the unit sphere $S^{d-1}\subset\mathbb{R}^d$. This can be seen by localizing the above restriction inequality, i.e. assuming f is supported in a ball of radius N, and by using the uncertainty principle, which allows us to assume that $\hat{f}$ is essentially constant on squares of size $1/N$. The relevant sets A are of the form

$$A = \{Q(\frac{n_1}{Q_1} + \cdots + \frac{n_d}{Q_d}) \mid 0 < n_i \in \mathbb{Z},\ N^2 < n_1^2 + \cdots + n_d^2 \leq (N+1)^2\ \},$$

where $Q = Q_1 \cdots Q_d$ and the Q_i's are relatively prime integers of order N (see [M] for those matters.) The main objective of the paper is to show that random sets of integers $A \subset [1, N]$ of size N^ρ which are obtained by selecting each integer $1 \leq n \leq N$ with equal probability satisfy for all $\gamma > 0$

$$\sup_{|a_n| \leq 1} \Big\| \sum_{n \in A} a_n\ e(n\cdot) \Big\|_p \leq\ C_\gamma N^\gamma\ \Big\| \sum_{n \in A} e(n\cdot) \Big\|_p \tag{1.3}$$

with large probability (see Theorem 4.4). Theorem 4.4 relies on a nice argument developed by Bourgain in [B4]. In section (4.3) we first provide a method for proving a weaker variant of Theorem 4.4 (see Proposition 4.6) which will later allow us to extend this result for certain values of p by showing a that the N^γ-term is not necessary (see Theorem 4.12). For this we rely on a probabilistic lemma from Bourgain's work [B1].

In addition to random subsets in the last section we also consider perturbations of arithmetic progressions. This means that each element of a given arithmetic progression is shifted independently and randomly by some small amount. We again show that most sets obtained in this fashion satisfy (1.3) for any $\gamma > 0$, see Theorem 5.6. As before, the method can be presented abstractly for perturbations of arbitrary sets A that satisfy condition (2.3). Given the fact that even a single explizit frequency set A of satisfying e.g. $|A \cap [0, N]| \approx N^\alpha, 0 < \alpha < 1$, as well as e.g inequality (4.15) is not known, we think that this is worth mentioning.

Acknowledgment: The authors would like to thank Jean Bourgain for several suggestions and comments. The first author was supported NSF grant DMS-0300416. The second author was partially supported by an NSF grant, DMS-0070538, and a Sloan fellowship.

2 Some generalities

In order to justify the size restriction $|A| \asymp N^\rho, 0 < \rho < 1$, on a frequency set $A \subset [1, N]$ we remark that by Hausdorff-Young's inequality one always has the bound

$$\sup_{|a_n| \leq 1} \Big\| \sum_{n \in A} a_n e(n\cdot) \Big\|_p \leq C \Big(\frac{N}{|A|}\Big)^{\frac{1}{p}} \Big\| \sum_{n \in A} e(n\cdot) \Big\|_p.$$

Together with the the obvious lower bound $\Big\| \sum_{n \in A} e(n\cdot) \Big\|_p^p \gtrsim |A|^p N^{-1}$ this settles the case of any large sets A, i.e. $\rho = 1$, as well as all arithmetic progressions. Another easy estimate can be obtained by interpolation. Indeed, if $2 < p < 4$, say, then interpolating between 2 and 4 yields the bound $C = O(N^\gamma), \gamma \leq (1 - \frac{p}{4})(1 - \frac{2}{p})$. It turns out that this interpolation can be done more carefully, which gives optimal results for sets A whose Dirichlet kernel satisfies a certain "reverse interpolation inequality." To this end, consider the convex set of trigonometric polynomials given by

$\mathcal{P}_A := \{\sum_{n\in A} a_n e(n\theta) \mid |a_n| \le 1\}$. Then for any odd integer $p > 2$,

$$\sup_{|a_n|\le 1} \int_0^1 \Big|\sum_{n\in A} a_n\, e(n\theta)\Big|^p d\theta = \sup_{|a_n|\le 1} \sum_{n\in A} a_n \int_0^1 e(n\theta) \sum_{k\in A} \bar{a}_k\, e(-k\theta) \Big|\sum_{\ell\in A} a_\ell\, e(\ell\theta)\Big|^{p-2} d\theta$$

$$\le \sup_{g\in\mathcal{P}_A} \sqrt{|A|} \left(\sum_{n\in A} \big|\, \widehat{g|g|^{p-2}}(n)\, \big|^2\right)^{\frac12} \le \sup_{g\in\mathcal{P}_A} \sqrt{|A|}\ \|g\|_{2(p-1)}^{p-1} \tag{2.1}$$

$$\le \Big\|\sum_{n\in A} e(n\cdot)\Big\|_2 \ \Big\|\sum_{n\in A} e(n\cdot)\Big\|_{2(p-1)}^{p-1}. \tag{2.2}$$

Here the first inequality sign in (2.1) follows by putting absolute values inside and Cauchy-Schwarz, the second is Plancherel, and (2.2) uses the majorant property on $2(p-1) \in 2\mathbb{N}$. Now assume for all $\varepsilon > 0$ the following condition

$$\Big\|\sum_{n\in A} e(n\cdot)\Big\|_2 \ \Big\|\sum_{n\in A} e(n\cdot)\Big\|_{2(p-1)}^{p-1} \le C_\varepsilon\, N^\varepsilon \ \Big\|\sum_{n\in A} e(n\cdot)\Big\|_p^p, \tag{2.3}$$

with C_ε depending only on ε. In view of the preceding, one then has (1.3) for any $\gamma > 0$. This condition, which is of basic importance for most of our work, is basically the reverse of the usual interpolation inequality. One checks immediately that arithmetic progressions satisfy (2.3). Also, observe that any frequency set A for which (2.3) holds for all p satisfies (1.3) for all p with $\gamma > 0$. Indeed, this follows inductively from the argument leading up to (2.2) using the majorant property from the previous stage $2(p-1)$ to pass to the next stage p. Finally, interpolation is required to obtain the desired bound for all p (at the cost of N^ε). Another case which is covered by this argument, but not the previous one based on Hausdorff-Young, are multi-dimensional arithmetic progressions. For example, one easily checks that

$$A = \{b + j_1 a_1 + j_2 a_2 \mid 0 \le j_1 < L_1,\ 0 \le j_2 < L_2\} \tag{2.4}$$

with $a_1 L_1 < a_2$, satisfies

$$\Big\|\sum_{n\in A} e(n\cdot)\Big\|_p^p \asymp (L_1 L_2)^{p-1}$$

for $p > 1$. Another interesting case are the squares $A = \{n^2 \mid 1 \le n \le \sqrt{N}\}$. In this case it is well-known that the there is a "kink" at $p = 4$ (see e.g. [B3]),

$$\begin{aligned} \Big\|\sum_{n\in A} e(n\cdot)\Big\|_p &\le C_\varepsilon\, N^{\varepsilon+\frac12} \ \text{ if } 2 \le p \le 4, \\ \Big\|\sum_{n\in A} e(n\cdot)\Big\|_p &\le C_\varepsilon\, N^{1-\frac{2}{p}+\varepsilon} \ \text{ if } p \ge 4, \end{aligned}$$

so that (2.3) holds only for $2 \le p \le 3$. In particular, the argument leading up to (2.2) gives the (trivial) statement that the majorant property holds at $p = 3$ for the squares. A nontrivial statement can be obtained by improving on the use of Plancherel in (2.1). Indeed, it is a well-known fact that

$$\Big\|\sum_{n=1}^N a_n e(n^2\theta)\Big\|_4 \le C_\varepsilon\, N^\varepsilon \left(\sum_{n=1}^N |a_n|^2\right)^{\frac12} \Longleftrightarrow \left(\sum_{n=1}^N |\hat{f}(n^2)|^2\right)^{\frac12} \le C_\varepsilon\, N^\varepsilon \|f\|_{L^{\frac43}(\mathbb{T})}, \tag{2.5}$$

the second statement being the dual of the first. This can be checked by reducing the L^4-norm to an L^2-norm by squaring, and then using Cauchy-Schwarz and the N^ε-bound on the divisor function, see [B3]. We now repeat the argument leading up to (2.2) to conclude the following. Let

$$\mathcal{P} := \{\sum_{n=1}^{N} a_n e(n^2\theta) \mid |a_n| \le 1\}.$$

If $p = 3k+1$, then one can apply the majorant property at $\frac{4}{3}(p-1)$ so that

$$\begin{aligned}
&\sup_{|a_n|\le 1} \int_0^1 \Big|\sum_{n=1}^{N} a_n e(n^2\theta)\Big|^p \, d\theta = \sup_{|a_n|\le 1} \sum_{n=1}^{N} a_n \int_0^1 e(n^2\theta) \sum_{k=1}^{N} \bar{a}_k e(-k^2\theta) \Big|\sum_{\ell=1}^{N} a_\ell e(\ell^2\theta)\Big|^{p-2} \, d\theta \\
(2.6)\qquad &\le \sup_{g\in\mathcal{P}} \sqrt{|A|} \left(\sum_{n=1}^{N} \big|\widehat{g|g|^{p-2}}(n^2)\big|^2\right)^{\frac12} \le \sup_{g\in\mathcal{P}} \sqrt{|A|}\; \|g\|_{\frac43(p-1)}^{p-1} \\
&\le \Big\|\sum_{n=1}^{N} e(n^2\cdot)\Big\|_2 \Big\|\sum_{n=1}^{N} e(n^2\cdot)\Big\|_{\frac43(p-1)}^{p-1} \le C_\varepsilon\, N^\varepsilon N^{\frac12} N^{p-\frac52} \le C_\varepsilon\, N^\varepsilon N^{p-2} \\
&\le C_\varepsilon\, N^\varepsilon \Big\|\sum_{n=1}^{N} e(n^2\cdot)\Big\|_p^p.
\end{aligned}$$

Here we used (2.5) in (2.6). This implies that for the sequence of squares (1.3) holds with any $\gamma > 0$ at $p = 7, 13, 19$ etc.

Another case of sets A that <u>do not</u> satisfy (2.3) are random subsets $A \subset [1, N]$. Indeed, we show below that random sets A which are obtained by selecting each integer $1 \le n \le N$ with probability τ have the property that for $p > 1$

$$\mathbb{E}\Big\|\sum_{n\in A} e(n\theta)\Big\|_p^p \asymp \tau^p N^{p-1} + (\tau N)^{\frac{p}{2}},$$

see Proposition 4.6. The two terms on the right balance at $\tau_{crit} = N^{-1+\frac{2}{p}}$ so that it is clear that (2.3) cannot hold in general. The main objective of the following section is to show that nevertheless, such random subsets <u>do satisfy</u> (1.3) with large probability. The method to some extent resembles the calculation from (2.2), but is of course more involved. We rely on a probabilistic lemma from Bourgain's work [B1].

It is possible to abstract the arguments below, and then verify that various examples satisfy the conditions of such an abstract theorem, the most important one being condition (2.3). More precisely, starting with a *deterministic* set A, define $\mathcal{S}_N(\omega) = \{n \in A \mid \xi_n = 1\}$ where ξ_n are i.i.d. selector variables satisfying $\mathbb{P}[\xi_n = 1] = \tau = 1 - \mathbb{P}[\xi_n = 0]$. If, amongst other things, (2.3) holds for A, then much of what is done in the following section goes through. On the other hand, some improvements which we obtain below for the case of arithmetic progressions are not easily axiomatized. Moreover, since we do not have any examples apart from (multi-dimensional) arithmetic progressions, we have decided against casting this into a more general framework. Thus, we write out the main argument only for arithmetic progressions. If (2.3) is violated, then our method applies only to certain p or after suitable modifications. For example, one can check that the machinery which we develop below shows that with high probability random subset of the squares satisfy (1.3) at $p = 7$ for any $\gamma > 0$. This requires invoking the (almost) $\Lambda(4)$ property of the squares as in (2.6). It seems difficult to obtain the desired bound for all p in case of the squares.

3 Failure of the majorant property

In order to establish a frequency set A for which the constant in the majorant inequality (1.1) is growing by a power in N we will need the following

Lemma 3.1. *Suppose $p > 2$ is not an even integer, then there are trigonometric polynomials q and Q with with coefficients in $\{0, 1, -1\}$ such that $|\hat{q}(n)| = \hat{Q}(n)$ and*

$$\|q(e^{2\pi it})\|_p > (1 + \delta_p)\ \|Q(e^{2\pi it})\|_p.$$

Proof. For $m, k \in \mathbb{N}$ define polynomials q and Q as follows

$$q(z) = (1 + z^k)(1 - z^m) \qquad \text{and} \qquad Q(z) = (1 + z^k)(1 + z^m),$$

where $z = e(t)$. Let c_n be the Fourier coefficients of $f(t) = |\sin \pi t|^p$ and define $a_n(p) = \frac{1}{\pi}\int_0^\pi (\sin t)^p\, e^{-int}\, dt$, which satisfies the following recurrence formulae:

$$2ia_n(p) = a_{n-1}(p-1) - a_{n+1}(p-1) \qquad \text{and} \qquad a_{n-1}(p-1) = i(1 + \frac{n}{p})\ a_n(p).$$

Since $c_n = a_{2n}(p)$ a little algebra gives

$$c_{n+1} = \frac{n - \alpha}{n + 1 + \alpha}\ c_n \qquad \text{where} \quad \alpha = p/2, \tag{3.1}$$

and $c_n = \bar{c}_n = c_{-n}$. Note that for $F(t) = |\cos \pi t|^p$ we have $\widehat{F}(n) = (-1)^n c_n$. By using Plancherel's identity and by choosing m, k relatively prime we get

$$\begin{aligned}
\|q(e^{2\pi it})\|_p^p - \|Q(e^{2\pi it})\|_p^p &= 4^p \int_0^1 F(kt)\ f(mt) - F(kt)\ F(mt)\ dt \\
&= 4^p \sum_{n\in\mathbb{Z}} (\ (-1)^{kn}\ c_{nm}\ c_{nk} - (-1)^{mn+kn}\ c_{nm}\ c_{nk}\ ) \\
&= 4^p \sum_{n\in\mathbb{Z}} (-1)^{nk}\ c_{nm}\ c_{nk}\ (1 - (-1)^{nm})
\end{aligned}$$

We choose k even and $m = k + 1$. Hence only odd n contribute to the latter sum which evaluates to

$$4^{p+1} \sum_{n\geq 1, n \text{ odd}} c_{nm}\ c_{nk}.$$

By the recursion formula for c_n we see that if $k, m > p/2$ both term in the sum have the same sign. The lemma follows. ∎

Theorem 3.2. *Suppose $p > 2$ is an not even integer and N a sufficiently large integer. Then there exist $\alpha_p > 0$, a frequency set $E \subset [0, N] \cap \mathbb{Z}$ and a sequence $\epsilon_j \in \{-1, 1\}$ such that*

$$\| \sum_{n \in E} \epsilon_j \; e^{i2\pi nx} \|_p \geq N^{\alpha_p} \; \| \sum_{n \in E} e^{i2\pi nx} \|_p \tag{3.2}$$

Proof. The previous Lemma provides a trigonometric polynomial q of degree $d > 1$ with Fourier coefficients in $\{0, -1, 1\}$ whose majorant Q satisfies $|\widehat{q}(l)| = \widehat{Q}(l)$, for $l \in \mathbb{Z}$, and

$$\|q\|_p \geq (1 + \delta) \; \|Q\|_p. \tag{3.3}$$

for some $\delta > 0$. We will inductively construct a finite Riesz product $q_k(x) = \prod_{j=0}^{k} \; q(m_j x)$ where $m_j \in \mathbb{Z}$ are randomly chosen in the interval $[M^j, 2M^j]$. Note that by choosing $M > d^p$ sufficiently large the Fourier coefficients of q_k are again contained in $\{0, 1, -1\}$. We claim that for the majorant Q_k with $\hat{Q}_k(n) = |\widehat{q}_k(n)|$ we have

$$\|q_k\|_p^p \geq (1 + \delta)^k \; \|Q_k\|_p^p. \tag{3.4}$$

This gives us (3.2) since q_k is of degree at most $N \leq 2dM^k$. Inequality (3.4) will be shown inductively: Define

$$f_k(x) = |q_k(x)|^p \qquad \text{and} \qquad g_k(x) = |q(m_{k+1}\, x)|^p.$$

Note that g_k has frequencies in $m_{k+1}\mathbb{Z}$. By Plancherel's identity we obtain for $T > 0$

$$\begin{aligned} \|q_{k+1}\|_p^p = \int_0^1 f_k(x)\, g_k(x)\, dx &= \widehat{f_k}(0) \; \widehat{g_k}(0) + \sum_{0<|l|<T} \widehat{f_k}(m_{k+1}l) \; \widehat{g_k}(-m_{k+1}l) + \sum_{|l|\geq T} \widehat{f_k}(m_{k+1}l) \; \widehat{g_k}(-m_{k+1}l) \\ &= \qquad A_k \qquad\qquad + \qquad\qquad B_k \qquad + \qquad C_k \end{aligned}$$

We have by induction

$$A_k = \|q_k\|_p^p \; \|q\|_p^p \geq (1 + \delta)^{k+p} \; \|Q_k\|_p^p \; \|Q\|_p^p.$$

To estimate C_k, note that $F = |q|^p$ is at least twice differentiable. Therefore $|n^2 \, \widehat{F}(n)| \leq c_1$, with c_1 depending only on d and p, hence $|\widehat{g}_k(-m_{k+1}l)| \leq c_1/l^2$ and by Cauchy-Schwarz and Parseval's identity we get

$$C_k^2 \leq \frac{c_2}{T^3} \sum_{|l|>T} |\widehat{f}_k(m_{k+1}l)|^2 \leq \frac{c_2}{T^3} \int_0^1 |q_k(x)|^{2p} \; dx \leq \frac{c_2 \; d^{2pk}}{T^3},$$

where we used $\|q_k\|_\infty \leq d^k$. To estimate B_k we apply Cauchy-Schwarz

$$B_k^2 \leq c_3^2 \sum_{0<|l|<T} |\widehat{f}_k(m_{k+1}l)|^2,$$

where c_3 is the L^2-norm of F, i.e. only dependent on d and p. We will need to specify $n = m_{k+1}$. To do this, let I be the set of integers in $[M^{k+1}, 2M^{k+1}]$. Then with $B_k = B_k(n)$

$$\frac{1}{|I|} \sum_{n \in I} \; |B_k(n)|^2 \leq c_3^2 \; \frac{1}{|I|} \sum_{n \in I, 0<|l|<T} |\widehat{f}_k(nl)|^2 \leq c_3^2 \; \frac{1}{|I|} \sum_{0<m<2TM^{k+1}} d(m) \; |\widehat{f}_k(m)|^2$$

with $R = 2TM^{k+1}$ and $d(m)$ is the number of divisors of m, which is at most of order $e^{c \log R / \log\log R} \leq c_\epsilon R^\epsilon$. By choosing $T = M^{100pk}$ and since $|I| = M^{k+1}$ we may bound the later term by

$$\frac{c_3^2 \, c_{\epsilon'}}{M^{k+1}} \; M^{\epsilon' k} \; \|f_k\|_2^2$$

for all $\epsilon' > 0$. Since

$$\|f_k\|_2^2 = \int_0^1 |q_k(x)|^{2p} \, dx \leq d^{2kp}, \tag{3.5}$$

by *pigeonholing* we find $n \in I$ such that

$$B_k \leq \tilde{c}_{\epsilon'} \left(\frac{d^{2p}}{M} \right)^{\frac{k}{2}} \; M^{\epsilon k} \leq M^{-k/4} \tag{3.6}$$

provided we chose M sufficiently large. By collecting the estimates for A_k, B_k and C_k by adjusting M (to absorb c_2) we get:

$$\|q_{k+1}\|_p^p \geq (1+\delta)^{k+p} \; \|Q_k\|_p^p \; \|Q\|_p^p - 2M^{-k/4} \tag{3.7}$$

$$\geq (1+\delta)^{k+p} \; (1 - o(1)) \; \|Q_k\|_p^p \; \|Q\|_p^p, \tag{3.8}$$

where the o-term is refers to $M \to \infty$. We can perform the same analysis for Q_{k+1}. We only need to possibly modify the choice of m_{k+1}. However, since (3.5) holds for q replaced by Q (in the definition of f and g) we can choose m_{k+1} such that, say, the sum of the moduli of the B_k-term's for q_{k+1} and Q_{k+1} satisfy the above bound as well. Hence,

$$\|Q_{k+1}\|_p^p \leq (1 + o(1)) \; \|Q_k\|_p^p \; \|Q\|_p^p$$

and therefore

$$\|q_{k+1}\|_p^p \geq (1+\delta)^{k+p} \; (1 - o(1)) \; \|Q_k\|_p^p \; \|Q\|_p^p \geq (1+\delta)^k \; \|Q_{k+1}\|_p^p.$$

□

4 Random subsets have the majorant property

4.1 Random sums over asymmetric Bernoulli variables

We first dispense with some simple technical statements about the behavior of random sums with asymmetric Bernoulli variables as summands. They are definitely standard, but lacking a precise reference we prefer to present them.

Lemma 4.1. *Let η_j be i.i.d. variables so that $\mathbb{P}[\eta_j = 1-\tau] = \tau$, $\mathbb{P}[\eta_j = -\tau] = 1-\tau$. Here $0 < \tau < 1$ is arbitrary. Let $N \geq 1$ and $\{a_j\}_{j=1}^N \in \mathbb{C}$ be given. Define $\sigma^2 = \tau(1-\tau)\sum_{j=1}^N |a_j|^2$. Then for $\lambda > 0$,*

$$\mathbb{P}\Big[\Big| \sum_{j=1}^N a_j \eta_j \Big| > \lambda\sigma \Big] \leq 4e^{-\frac{\lambda^2}{8}}$$

provided

$$\max_{1\le j\le N} \lambda|a_j| \le 4\sigma. \tag{4.1}$$

Proof. Assume first that all $a_j \in \mathbb{R}$. Then for any $t > 0$

$$\mathbb{P}\Big[\sum_{j=1}^N a_j\eta_j > \lambda\sigma\Big] \le e^{-t\lambda\sigma}\,\mathbb{E}\exp\Big(t\sum_{j=1}^N a_j\eta_j\Big) \tag{4.2}$$

$$= e^{-t\lambda\sigma}\prod_{j=1}^N \Big[\tau e^{(1-\tau)ta_j} + (1-\tau)e^{-\tau t a_j}\Big] \tag{4.3}$$

Next, we claim that

$$\tau e^{(1-\tau)x} + (1-\tau)e^{-\tau x} \le \exp(2\tau(1-\tau)x^2) \quad \text{for all} \quad |x|\le 1. \tag{4.4}$$

Observe that this property fails for $x = \tau^{-\frac{1}{2}}$. To prove this, set

$$\phi_\tau(x) = \exp(2\tau(1-\tau)x^2) - \tau e^{(1-\tau)x} - (1-\tau)e^{-\tau x}.$$

By symmetry it suffices to consider the case $0\le x\le 1$ and to show that $\phi_\tau \ge 0$ there. Clearly,

$$\begin{aligned} \phi_\tau'(x) &= \tau(1-\tau)\big[4x\exp(2\tau(1-\tau)x^2) - e^{(1-\tau)x} + e^{-\tau x}\big] \\ &\ge \tau(1-\tau)\big[4x - e^{(1-\tau)x} + e^{-\tau x}\big] \end{aligned} \tag{4.5}$$

Differentiating the expression in brackets yields

$$4 - (1-\tau)e^{(1-\tau)x} - \tau e^{-\tau x} \ge 4 - (1-\tau)e^{(1-\tau)x} - \tau e^{(1-\tau)x} \ge 4 - e > 0$$

for all $0\le x\le 1$. It follows that $\phi_\tau'(x)\ge 0$ for $0\le x\le 1$ and since $\phi_\tau(0)=0$ we also have $\phi_\tau(x)\ge 0$ for $0\le x\le 1$, as desired. Inserting (4.4) into (4.3) gives

$$\mathbb{P}\Big[\sum_{j=1}^N a_j\eta_j > \lambda\sigma\Big] \le \min_{t>0} e^{-t\lambda\sigma}\exp(2t^2\sigma^2) = e^{-\frac{\lambda^2}{8}}$$

provided for the minimizing choice of $t = t_0$ one has $\max_j |t_0\, a_j| \le 1$. But $t_0 = \frac{\lambda}{4\sigma}$ and this condition therefore reads

$$\max_{1\le j\le N} \frac{|\lambda||a_j|}{4\sigma} \le 1,$$

which is precisely (4.1). Evidently, the same bound also holds for deviations less than $-\lambda\sigma$, which gives $2e^{-\lambda^2/8}$ as an upper bound on the large deviation probability in the real case. Finally, if $a_n \in \mathbb{C}$, then one splits into real and complex parts. □

Lemma 4.1 immediately leads to the following version of the Salem–Zygmund inequality for asymmetric variables.

Corollary 4.2. *With η_n and σ as in the previous lemma*

$$\mathbb{P}\Big[\sup_{\theta\in\mathbb{T}}\Big|\sum_{n=1}^{N} a_n\,\eta_n\,e(n\theta)\Big| > 20\,\sigma\sqrt{\log N}\Big] \le 4N^{-8}$$

for any $a_n \in \mathbb{C}$ provided the following conditions hold:

$$\begin{aligned}\sup_{1\le n\le N} 10|a_n|^2\log N &\le \sigma^2 = \tau(1-\tau)\sum_{k=1}^{N}|a_k|^2 \\ 10 &\le \tau(1-\tau)N\log N. \end{aligned} \tag{4.6}$$

Proof. Let $\{\theta_j\}_{j=1}^{N^2} \subset \mathbb{T}$ be a N^{-2}-net. Denote

$$T_{N,\omega}(\theta) := \sum_{n=1}^{N} a_n\,\eta_n(\omega)\,e(n\theta).$$

By using $T'_{N,\omega}(\theta) = T_{N,\omega} * D'_N(\theta)$, where D_N denotes the Dirichlet kernel, Cauchy-Schwarz and Parseval's identity gives

$$\begin{aligned}&\min_j |T_{N,\omega}(\theta) - T_{N,\omega}(\theta_j)| \le N^{-2}\|T'_{N,\omega}\|_\infty \\ &\le N^{-2}\|T_{N,\omega}\|_2\|D'_N\|_2 \le N^{-2}\Big(\sum_{n=1}^{N}|a_n|^2\Big)^{\frac{1}{2}}\,2N^{\frac{3}{2}} \\ &= \frac{2\sigma N^{-\frac{1}{2}}}{\sqrt{\tau(1-\tau)}} \le 10\sigma\sqrt{\log N}.\end{aligned}$$

The final inequality here follows from our assumption (4.6). Therefore, by Lemma 4.1,

$$\begin{aligned}\mathbb{P}\Big[\sup_{\theta\in\mathbb{T}}\Big|\sum_{n=1}^{N} a_n\,\eta_n\,e(n\theta)\Big| > 20\,\sigma\sqrt{\log N}\Big] &\le \sum_{j=1}^{N^2}\mathbb{P}\Big[\Big|\sum_{n=1}^{N} a_n\,\eta_n\,e(n\theta_j)\Big| > 10\,\sigma\sqrt{\log N}\Big] \\ &\le 4N^2\exp(-100\log N/8) \le 4N^{-8},\end{aligned}$$

which is precisely the bound claimed in the lemma. The first condition in (4.6) ensures that (4.1) holds. □

In the proof of Theorem 4.4 and 4.12 we shall need to know the typical size of the easier norm in (4.18). We determine this norm in the following lemma.

Lemma 4.3. *Let ξ_j be selector variables as above with $\tau = N^{-\delta}$, $0 < \delta < 1$ fixed. Let $p \ge 2$ and define*

$$I_{p,N}(\omega) = \int_0^1 \Big|\sum_{n=1}^{N}\xi_n(\omega)e(n\theta)\Big|^p\,d\theta.$$

Then for some constants C_p,

$$C_p^{-1}\left(\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}}\right) \le \mathbb{E}\, I_{p,N} \le C_p\left(\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}}\right).$$

Moreover, there is some small constant c_p *such that*

$$\mathbb{P}\Big[I_{p,N} \le c_p(\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}})\Big] \to 0$$

as $N \to \infty$.

Proof. Let $\eta_n(\omega) = \xi_n(\omega) - \tau$, so that $\mathbb{E}\,\eta_n = 0$ and $\mathbb{E}\,\eta_n^2 = \tau(1-\tau)$. Then

$$\begin{aligned} I_{p,N}(\omega) &\lesssim \int_0^1 \Big|\sum_{n=1}^N \tau e(n\theta)\Big|^p d\theta + \int_0^1 \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p d\theta \\ &\lesssim \tau^p N^{p-1} + \int_0^1 \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p d\theta. \qquad (4.7) \end{aligned}$$

One now checks that

$$\mathbb{E}\int_0^1 \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p d\theta \le C_p\,(N\tau(1-\tau))^{\frac{p}{2}}.$$

This can be verified by expanding the norm for even p and then interpolating. Indeed,

$$\begin{aligned} &\mathbb{E}\int_0^1 \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^{2k} d\theta = \mathbb{E}\int_0^1 \Big|\sum_{n_1,\dots,n_k=1}^N \eta_{n_1}\dots\eta_{n_k} e((n_1+\dots+n_k)\theta)\Big|^2 d\theta \\ &= \sum_n \mathbb{E}\Big|\sum_{n_1+\dots+n_k=n} \eta_{n_1}\dots\eta_{n_k}\Big|^2 = \sum_{n_1+\dots+n_k=m_1+\dots+m_k} \mathbb{E}\left[\eta_{n_1}\dots\eta_{n_k}\eta_{m_1}\dots\eta_{m_k}\right] \\ &\le C_k \sum_{r=1}^k \sum_{\substack{n_1,\dots,n_r=1 \\ s_1+\dots+s_r=2k,\ s_i\ge 2}}^N \mathbb{E}\,|\eta_{n_1}|^{s_1}\cdot\ldots\cdot\mathbb{E}\,|\eta_{n_r}|^{s_r} \qquad (4.8) \\ &\le C_k \sum_{r=1}^k N^r(\tau(1-\tau))^r \le C_k(N\tau(1-\tau))^k. \qquad (4.9) \end{aligned}$$

The constants in (4.8) and (4.9) are of a combinatorial nature and not necessarily the same. The relevant point in (4.8) is that $s_i \ge 2$ which is due to independence and $\mathbb{E}\,\eta_j = 0$. In particular, $s_i \ge 2$ implies the important fact $r \le k$. Moreover, to pass to the last line we used that for every positive integer $s \ge 2$

$$\tau(1-\tau) \ge \mathbb{E}\,\eta_j^s = \tau(1-\tau)(\tau^{s-1} + (1-\tau)^{s-1}) \ge 2^{2-s}\tau(1-\tau).$$

To obtain the lower bound on the expectation, one splits the integral in θ into the region where the Dirichlet kernel dominates the mean zero random sum and vice versa. More precisely, with

$h = \sqrt{\tau N^{-1}} = N^{-\frac{1+\delta}{2}}$,

$$\begin{aligned} I_{p,N} \ \gtrsim\ & \int_{|\theta|<\frac{1}{N}} \Big|\sum_{n=1}^{N} \tau e(n\theta)\Big|^p d\theta - \int_{|\theta|<\frac{1}{N}} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta \\ & + \int_h^{1-h} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta - \int_h^{1-h} \Big|\sum_{n=1}^{N} \tau e(n\theta)\Big|^p d\theta \\ \gtrsim\ & \tau^p N^{p-1} - C \int_{|\theta|<\frac{1}{N}} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta + \int_{|\theta|>h} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta - C\tau^p\, h^{1-p}. \end{aligned} \tag{4.10}$$

According to Corollary 4.2, the first integral in (4.10) is

$$\lesssim N^{-1}(\log N)^{\frac{p}{2}} (\tau(1-\tau)N)^{\frac{p}{2}} \tag{4.11}$$

up to a negligible probability. For the second, one has because of $p \geq 2$

$$\begin{aligned} \int_h^{1-h} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta \ \geq\ & \int_0^1 \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta - \int_{|\theta|\leq h} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta \\ \geq\ & \left(\int_0^1 \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^2 d\theta\right)^{\frac{p}{2}} - \int_{|\theta|\leq h} \Big|\sum_{n=1}^{N} \eta_n e(n\theta)\Big|^p d\theta \\ \geq\ & \left(\sum_{n=1}^{N} \eta_n^2\right)^{\frac{p}{2}} - C\, h\, (N\tau\, \log N)^{\frac{p}{2}} \end{aligned} \tag{4.12}$$

where the last term in (4.12) is obtained from Corollary 4.2. Using $p \geq 2$ again,

$$\mathbb{E}\left(\sum_{n=1}^{N} \eta_n^2\right)^{\frac{p}{2}} \geq \left(\mathbb{E}\sum_{n=1}^{N} \eta_n^2\right)^{\frac{p}{2}} \geq \Big(N\tau(1-\tau)\Big)^{\frac{p}{2}}.$$

In fact, Lemma 4.1 gives the following more precise estimate:

$$\mathbb{P}\Big[\Big|\sum_{n=1}^{N} (\eta_n^2 - \mathbb{E}\,\eta_n^2)\Big| \geq \lambda\sqrt{N\,\mathbb{E}\,(|\eta_1^2 - \mathbb{E}\,\eta_1^2|^2)}\Big] \leq 4e^{-\lambda^2/8} \tag{4.13}$$

provided the conditions (4.1) hold. One checks that $\mathbb{E}\,(|\eta_1^2 - \mathbb{E}\,\eta_1^2|^2) \asymp \tau(1-\tau)$. Hence it follows from (4.13) that for large N

$$\begin{aligned} & \mathbb{P}\Big[\sum_{n=1}^{N} \eta_n^2 \leq \frac{1}{2}\,\mathbb{E}\sum_{n=1}^{N} \eta_n^2 = \frac{1}{2} N\tau(1-\tau)\Big] \leq \mathbb{P}\Big[\Big|\sum_{n=1}^{N} \eta_n^2 - \mathbb{E}\sum_{n=1}^{N} \eta_n^2\Big| \geq \frac{1}{2} N\tau(1-\tau)\Big] \\ & \leq \mathbb{P}\Big[\Big|\sum_{n=1}^{N} \eta_n^2 - \mathbb{E}\sum_{n=1}^{N} \eta_n^2\Big| \geq \log N \sqrt{N\tau(1-\tau)}\Big] \leq 4e^{-(\log N)^2/8}, \end{aligned}$$

since with our choice of parameters (4.1) hold for large N. Inserting this bound into (4.12) now yields (recall that $h = \sqrt{\tau N^{-1}} = N^{-\frac{1+\delta}{2}}$)

$$\int_h^{1-h} \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p \, d\theta \geq \Big(\frac{1}{2} N\tau(1-\tau)\Big)^{\frac{p}{2}} - C\, N^{-\frac{1+\delta}{2}} \, (N\tau \, \log N)^{\frac{p}{2}} \gtrsim (N\tau)^{\frac{p}{2}}$$

up to negligible probability. In view of this bound and (4.11), one obtains from (4.10) that

$$\begin{aligned} I_{p,N} &\gtrsim \tau^p N^{p-1} - C \int_{|\theta|<\frac{1}{N}} \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p \, d\theta + \int_{|\theta|>h} \Big|\sum_{n=1}^N \eta_n e(n\theta)\Big|^p \, d\theta - C\tau^p \, h^{1-p} \\ &\gtrsim \tau^p N^{p-1} + (N\tau)^{\frac{p}{2}} \end{aligned}$$

up to negligible probability. To remove the final term in the first line we used that $(N\tau)^{\frac{p}{2}} \gg \tau^p h^{1-p}$ which follows from our choice of h provided N is big. $\square$

4.2 Random sets satisfy the majorant inequality (1.3)

In this section we will show

Theorem 4.4. *Let $0 < \delta < 1$ be fixed. For every positive integer N we let $\xi_j = \xi_j(\omega)$ be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$ where $\tau = N^{-\delta}$. Define a random subset*

$$S(\omega) = \{j \in [1, N] \mid \xi_j(\omega) = 1\}.$$

Then for every $\varepsilon > 0$ and $p \geq 2$ one has

$$\mathbb{P}\Big[\sup_{|a_n|\leq 1} \Big\| \sum_{n\in S(\omega)} a_n e(n\theta)\Big\|_{L^p(\mathbb{T})} \geq N^{\varepsilon} \Big\| \sum_{n\in S(\omega)} e(n\theta)\Big\|_{L^p(\mathbb{T})}\Big] \to 0 \tag{4.14}$$

as $N \to \infty$.

The proof of Theorem 4.4 relies on Slepian's Lemma and ideas in Bourgain's paper [B4]. In the next section we will present a variant of Theorem 4.4 which requires additional assumptions on the exponent δ as well as on p. However, this second method will lead us later to remove the N^{ε}-term in (4.14) in certain cases, for example when $p = 3$. This improvement (which we believe should hold in general, i.e. for $2 < p \notin 2\mathbb{N}$) relies on a method developed in Bourgain's work on the solution of the $\Lambda(p)$ problem, see [B1] and [B2]. In fact, in this situation we can avoid several complications that arose in Bourgain's work. Notice that Theorem 4.4 is implied by Bourgain's existence theorem of $\Lambda(p)$ sets provided $\delta \geq 1 - \frac{2}{p}$, but not for $\delta < 1 - \frac{2}{p}$. Indeed, in the former case the random set S will typically have cardinality $N^{\frac{2}{p}}$ or smaller, and such sets were shown by Bourgain [B1] to be $\Lambda(p)$-sets with large probability.

Let $\xi_j = \xi_j(\omega), 1 \leq j \leq N$, be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1-\tau$ where $\tau \in (0, 1)$. Define a random subset

$$S(\omega) = \{j \in [1, N] \mid \xi_j(\omega) = 1\}.$$

We sometimes drop the argument ω and write simply S for $S(\omega)$. The following result is a *discretized* version of a result shown by Bourgain in [B4].

Proposition 4.5. *Let $M \in \mathbb{N}, M < N$, and $T(\theta) = \sum_{n \in S(\omega)} a_n e(n\theta)$ be a trigonometric polynomial with frequencies in $S(\omega)$. Then there exists $C > 0$ independent of M and N such that*

$$\mathbb{E}_\omega \sup_{|a_n| \le 1} \sup_{|I| \le M} \sqrt{\sum_{j \in I} |T(\frac{j}{N})|^2} \le \tau N + C\ M\ \log N, \tag{4.15}$$

where the second supremum is over all integer sets $I \subset [0, N)$ with $|I| \le M$. In particular, we have for c_1 sufficiently large

$$\mathbb{E}_\omega \sup_{|a_n| \le 1} \sqrt{\sum_{|T(\frac{j}{N})|^2 \ge c_1 \tau N \log N} |T(\frac{j}{N})|^2} \le C\ \tau N. \tag{4.16}$$

To see that (4.16) follows from (4.15), we choose $M = \tau N / \log N$ and note that for $c_1 > 0$ sufficiently large the integer set $X = \{j \in [0, N) \,|\, |T(\frac{j}{N})|^2 \ge c_1 \tau N \log N\}$ is of size at most M. Note that otherwise, X would contains a subset I of size $|I| = M$ for which (4.15) implies: $\sqrt{M\ c_1\, \tau N\ \log N} \le (C+1)\, \tau N$, i.e. $c_1 \le (C+1)^2$.

For convenience we will include below Bourgain's proof of Proposition 4.5. With the bound on the expected size for Dirichlet kernels in L^p given by Lemma 4.3 we are prepared to the

Proof of Theorem 4.4: By the Marcinkiewicz-Zygmund inequality, see [Zyg, *p.28*], the L^p-norm (for $p > 1$) of a trigonometric polynomial of degree N is comparable with the Riemann sum over N equidistant points, i.e. for $f(\theta) = \sum_{n \in S(\omega)} a_n e(n\theta)$ with $|a_n| \le 1$ we have

$$\| \sum_{n \in S(\omega)} a_n e(n\theta) \Big\|^p_{L^p(\mathbb{T})} \approx \frac{1}{N} \sum_{j=0}^{N-1} |f(\frac{j}{N})|^p$$

with hidden constants depending on p but independent of N. We divide the Riemann sum into $I = \{j \,|\, |f(\frac{j}{N})|^2 \ge c_1 \tau N \log N\}$ and its complement J in $[0, N) \cap \mathbb{Z}$. Fix $\alpha, \beta > 0$ with $\alpha(p-2) + \beta = p/2$. Since $\|f\|_\infty \le |S|$ and the expected size of S is τN we find $\Omega_N \subset \Omega$ with $\mathbb{P}(\Omega_N) \to 1$, as $N \to \infty$, such that $\|f\|_\infty \le (\log N)^\alpha \tau N$. Also, by (4.16) we find $\Omega'_N \subset \Omega_N$ with $\mathbb{P}(\Omega'_N) \to 1$ such that for $\omega \in \Omega'_N$ we have $\sum_{j \in I} |f(\frac{j}{N})|^2 \le (\log N)^\beta (\tau N)^2$. Hence, for $\omega \in \Omega'_N$, we get

$$\begin{aligned}
\frac{1}{N} \sum_{j=0}^{N-1} |f(\frac{j}{N})|^p &= \frac{1}{N} \sum_{j \in I} |f(\frac{j}{N})|^{p-2}\, |f(\frac{j}{N})|^2 \ + \ \frac{1}{N} \sum_{j \in J} |f(\frac{j}{N})|^p \\
&\le (\log N)^{\alpha(p-2)} (\tau N)^{p-2} \frac{1}{N} \sum_{j \in I} |f(\frac{j}{N})|^2 \ + \ (c_1 \tau N \log N)^{\frac{p}{2}} \\
&\le C\ (\log N)^{\frac{p}{2}} \left(\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}} \right)
\end{aligned}$$

Hence, by Lemma 4.3 we find

$$\| \sum_{n \in S(\omega)} a_n e(n\theta) \Big\|^p_{L^p(\mathbb{T})} \le C\ (\log N)^{\frac{p}{2}}\ \mathbb{E}\, I_{p,N}$$

and the theorem follows with a possibly smaller subset of Ω'_N whose probability still approaches 1 as $N \to \infty$. $\square$

Proof of Proposition 4.5. We need to show that the expectation

$$L := \mathbb{E}_\omega \sup_{\{|a_n|\le 1,\ I\}} \big(\sum_{m\in I} \big| \sum_{j=1}^N a_j\ \xi_j(\omega)\ e(jm/N)\, \big|^2 \big)^{1/2} \le C\ \tau N.$$

Here the supremum is over all sets I with $|I| \le \tau N/\log N$. By $\ell^2(I)$-duality we may express the left-hand side by

$$\mathbb{E}_\omega \sup_{\{|a_n|\le 1,\ I\}} \sup_{\|b\|_{\ell^2(I)}=1} \big| \sum_{j=1}^N a_j\ \xi_j(\omega) \sum_{m\in I} b_m e(jm/N)\, \big| = \mathbb{E}_\omega \sup_{I,\ \|b\|_{\ell^2(I)}=1} \sum_{j=1}^N \xi_j(\omega)\, \big| \sum_{m\in I} b_m e(jm/N)\, \big|.$$

Write $\xi_j = \eta_j + \tau$, i.e. the $\eta_j's$ have vanishing expectation. It follows that

$$L \le \tau \sup_{I,b} \sum_{j=1}^N \big| \sum_{m\in I} b_m e(jm/N)\, \big| \ +\ \mathbb{E}_\omega \sup_{I,b} \sum_{j=1}^N \eta_j(\omega)\ \big| \sum_{m\in I} b_m e(jm/N)\, \big| =: L_1\ +\ L_2$$

By using the (ℓ^1, ℓ^∞)-duality and Cauchy-Schwarz the term L_1 is bounded by

$$\tau \sup_{b,\ I} \sup_{|c_j|\le 1} \big| \sum_{j=1}^N c_j \sum_{m\in I} b_m\ e(jm/N)\, \big| \le \tau \sup_{|c_j|\le 1} \Big(\sum_{m=1}^N \big| \sum_{j=1}^N c_j\ e(jm/N)\, \big|^2 \Big)^{1/2}.$$

So, Parseval's identity gives $L_1^2 \le \tau^2 N \sup_{|c_j|\le 1} \sum_{k=1}^N |c_k|^2 \le (\tau N)^2$. To bound the term L_2 we first note that for each choice of $\varepsilon_k = \pm 1$ and each bounded sequence of complex-valued functions $A_k(t)$ one has

$$\mathbb{E}_\omega \sup_t \big| \sum_{k=1}^N \eta_k(\omega)\ |A_k(t)| \big| \le 2\, \mathbb{E}_\omega \sup_t \big| \sum_{k=1}^N \varepsilon_k\ \eta_k(\omega)\ |A_k(t)| \big| \tag{4.17}$$

To see this, set $X = \{k \,|\, \varepsilon_k = 1\}$ and $Y = X^c$, the complement of X. Then

$$\mathbb{E}_\omega \sup_t \big| \sum_{k=1}^N \eta_k(\omega)\ |A_k(t)| \big| \le \mathbb{E}_\omega \sup_t \big| \sum_{k\in X} \eta_k(\omega)\ |A_k(t)| \big|\ +\ \mathbb{E}_\omega \sup_t \big| \sum_{k\in Y} \eta_k(\omega)\ |A_k(t)| \big|$$

Since $\mathbb{E}_{\omega'} \eta_k = 0$ we may rewrite the first term as

$$\mathbb{E}_\omega \sup_t \big| \mathbb{E}_{\omega'} \big(\sum_{k\in X} \varepsilon_k\ \eta_k(\omega)\ |A_k(t)| + \sum_{k\in Y} \varepsilon_k\ \eta_k(\omega')\ |A_k(t)|\ \big)\, \big|,$$

which is bounded by

$$\mathbb{E}_\omega \mathbb{E}_{\omega'} \sup_t \big| \sum_{k\in X} \varepsilon_k\ \eta_k(\omega)\ |A_k(t)|\ +\ \sum_{k\in Y} \varepsilon_k\ \eta_k(\omega')\ |A_k(t)|\, \big| = \mathbb{E}_\omega \sup_t \big| \sum_{k=1}^N \varepsilon_k\ \eta_k(\omega)\ |A_k(t)| \big|,$$

where we used independence. Exchanging X and Y the second term is seen to be bounded by the same expression, i.e. (4.17) holds. Since (4.17) remains true if we average over ε_k we obtain for L_2 the bound

$$L_2 \leq 2\,\mathbb{E}_{\varepsilon}\,\mathbb{E}_{\omega}\ \sup_{I,b} \Big| \sum_{j=1}^{N} \varepsilon_j\ \eta_j(\omega)\ \Big| \sum_{m\in I} b_m e(jm/N)\Big|\Big|.$$

We may now employ the contraction principle (see [T, p.222]) to majorize the above Rademacher sequence ε_j by Gaussian random variables g_j, i.e. we have

$$L_2 \leq 2\,\mathbb{E}_{\omega'}\,\mathbb{E}_{\omega}\ \sup_{I,b} \Big| \sum_{j=1}^{N} g_j(\omega')\ \eta_j(\omega)\ \Big| \sum_{m\in I} b_m e(jm/N)\Big|\Big|$$

By Slepian's Lemma (see [T, p.222]) for Gaussian processes we can bound right-hand side by

$$C\ \mathbb{E}_{\omega'}\,\mathbb{E}_{\omega}\ \sup_{I,b} \Big| \sum_{j=1}^{N} g_j(\omega')\ \eta_j(\omega)\ \sum_{m\in I} b_m e(jm/N)\Big|.$$

Hence, by evaluating the supremum over $\|b\|_{\ell^2(I)} = 1$ we find

$$\begin{aligned} L_2 \ &\leq\ C\ \mathbb{E}_{\omega'}\,\mathbb{E}_{\omega}\Big(\sum_{m\in I}\ \Big|\sum_{j=1}^{N} g_j(\omega')\ \eta_j(\omega)\ e(jm/N)\Big|^2\ \Big)^{1/2} \\ &\leq\ C\ \sup_{I}\sqrt{|I|}\ \ \mathbb{E}_{\omega}\,\mathbb{E}_{\omega'}\ \sup_{1\leq m\leq N}\Big|\sum_{j=1}^{N} g_j(\omega')\ \eta_j(\omega)\ e(jm/N)\Big| \end{aligned}$$

By the Salem-Zygmund's inequality [SZ] for Gaussian Fourier series we finally get

$$L_2 \leq C\ \big(\tau N/\log N\big)^{1/2}\ (\log N)^{1/2}\ \ \mathbb{E}_{\omega}\Big(\sum_{j=1}^{N}\eta_j(\omega)^2\Big)^{1/2} = C\ \tau N.$$

Hence $L \leq C\ \tau N$. □

4.3 Suprema of random processes

In this section we will first derive a proof of the following somewhat weaker version of Theorem 4.4.

Proposition 4.6. Let $0 < \delta < 1$ be fixed. For every positive integer N we let $\xi_j = \xi_j(\omega)$ be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$ where $\tau = N^{-\delta}$. Define a random subset

$$S(\omega) = \{j \in [1, N] \mid \xi_j(\omega) = 1\}.$$

Then for every $\varepsilon > 0$ and $4 \geq p \geq 2$ one has

$$\mathbb{P}\Big[\sup_{|a_n|\leq 1}\Big\|\sum_{n\in S(\omega)} a_n e(n\theta)\Big\|_{L^p(\mathbb{T})} \geq N^{\varepsilon}\Big\|\sum_{n\in S(\omega)} e(n\theta)\Big\|_{L^p(\mathbb{T})}\Big] \to 0 \tag{4.18}$$

as $N \to \infty$. Moreover, under the additional restriction $\delta \leq \frac{1}{2}$, (4.18) holds for all $p \geq 4$.

For a proof of Theorem 4.6 as well as for its improvements we now collect the statements from Bourgain's paper that we will need. The first is Lemma 1 from [B1] with $q_0 = 1$. In fact, Bourgain's lemma is slightly stronger because of certain $\log \frac{1}{\tau}$-factors. While these factors are important for his purposes, they play no role in our argument. We present the proof for the reader's convenience, following Bourgain's original argument. Another proof was found by Ledoux and Talagrand [LT] which is close to the ideology surrounding Dudley's theorem on suprema of Gaussian processes. While their point of view is perhaps more conceptual, we have found it advantageous to follow [B1]. Throughout, if $x \in \mathbb{R}^N$, then $|x| = |x|_{\ell^2_N} = \Big(\sum_{j=1}^N x_j^2 \Big)^{\frac{1}{2}}$ is the Euclidean norm. Secondly, $N_2(\mathcal{E}, t)$ refers to the L^2-entropy of the set $\mathcal{E}$ at scale t. Recall that this is defined to be the minimal number of L^2-balls of radius t needed to cover $\mathcal{E}$.

Lemma 4.7. *Let $\mathcal{E} \subset \mathbb{R}^N_+$, $B = \sup_{x\in\mathcal{E}} |x|$, and ξ_j be selector variables as above with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$, and $0 < \tau < 1$ arbitrary. Let $1 \le m \le N$. Then*

$$\mathbb{E} \sup_{x\in\mathcal{E}, |A|=m} \Big[\sum_{j\in A} \xi_j\, x_j \Big] \lesssim (\tau m + 1)^{\frac{1}{2}} B + \int_0^B \sqrt{\log N_2(\mathcal{E}, t)}\, dt$$

where N_2 refers to the L^2 entropy.

Proof. Let $\mathcal{E}_k$ be minimal 2^{-k}-nets for $\mathcal{E}$ with $2^{-k} \le B$. Let $B = 2^{-k_0}$. Then every $x \in \mathcal{E}$ can be written as

$$x = x_{k_0} + \sum_{k=k_0}^{\infty} (x_{k+1} - x_k) = x_{k_0} + \sum_{k=k_0}^{\infty} 2^{-k+1} y_k$$

where $x_k \in \mathcal{E}_k$ for every $k \ge k_0$. We can and do set $x_{k_0} = 0$. Now, $y_k \in \mathcal{F}_k$ where $\mathrm{diam}(\mathcal{F}_k) \le 1$ and $\#(\mathcal{F}_k) \le \#(\mathcal{E}_k) \cdot \#(\mathcal{E}_{k+1})$. Hence

$$\log \#\mathcal{F}_k \le C \log \#\mathcal{E}_{k+1}, \tag{4.19}$$

and thus

$$\mathbb{E} \sup_{x\in\mathcal{E}, |A|=m} \Big[\sum_{j\in A} \xi_j\, x_j \Big] \le \sum_{k\ge k_0} 2^{-k+1}\, \mathbb{E} \sup_{y\in\mathcal{F}_k,\, |A|\le m} \sum_{i\in A} \xi_i |y_i|. \tag{4.20}$$

Now fix some $k \ge k_0$ and write $\mathcal{F}$ instead of $\mathcal{F}_k$. Moreover, replacing every vector $y = \{y_j\}_{j=1}^N \in \mathcal{F}$ with the vector $\{|y_i|\}_{i=1}^N$, we may assume that $\mathcal{F} \subset \mathbb{R}^N_+$. Note that this changes neither the diameter nor the cardinality bound of $\mathcal{F}$. With $0 < \rho_1 < \rho_2$ to be determined, one has

$$\sum_{i\in A} \xi_i y_i \le \sum_{y_i \ge \rho_2} y_i + \sum_{i\in A,\, y_i\le\rho_1} y_i + \sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i \le \rho_2^{-1} \sum_{y_i\ge\rho_2} y_i^2 + m\rho_1 + \sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i.$$

Let $q = 1 + \lfloor \log \mathcal{F} \rfloor$. Since $|y| \le 1$, one concludes that

$$\mathbb{E} \sup_{y\in\mathcal{F},\,|A|\le m} \sum_{i\in A} \xi_i y_i \;\le\; \rho_2^{-1} + m\rho_1 + \mathbb{E} \sup_{y\in\mathcal{F}} \sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i$$

$$\lesssim\; \rho_2^{-1} + m\rho_1 + \mathbb{E}\left[\sum_{y\in\mathcal{F}}\Big(\sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i\Big)^q\right]^{\frac{1}{q}} \tag{4.21}$$

$$\lesssim\; \rho_2^{-1} + m\rho_1 + \left[\sum_{y\in\mathcal{F}} \mathbb{E}\Big(\sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i\Big)^q\right]^{\frac{1}{q}} \tag{4.22}$$

$$\lesssim\; \rho_2^{-1} + m\rho_1 + (\#\mathcal{F})^{\frac{1}{q}} \sup_{y\in\mathcal{F}} \left[\mathbb{E}\Big(\sum_{\rho_1<y_i<\rho_2} \xi_i\, y_i\Big)^q\right]^{\frac{1}{q}} \tag{4.23}$$

$$\lesssim\; \rho_2^{-1} + m\rho_1 + \sup_{|y|\le 1}\Big\|\sum_{\rho_1<y_i<\rho_2} \xi_i(\omega)\, y_i\Big\|_{L^q(\omega)}. \tag{4.24}$$

Here (4.21) follows from the embedding $\ell^q(\mathcal{F}) \hookrightarrow \ell^\infty(\mathcal{F})$, (4.22) follows from Hölder's inequality, and to pass from (4.23) to (4.24) one uses that

$$(\#\mathcal{F})^{\frac{1}{q}} = \exp[(\log \#\mathcal{F})/q] \le e$$

by our choice of $q = 1 + \lfloor \log \mathcal{F} \rfloor$. To control the last term in (4.24), we need the following simple estimate, see Lemma 2 in [B1]. By the multinomial theorem (for any positive integer q),

$$\begin{aligned}
\mathbb{E}\Big[\sum_{j=1}^n \xi_j\Big]^q &= \sum_{q_1+\ldots+q_n=q} \binom{q}{q_1,\ldots,q_n} \mathbb{E}\,\xi_1^{q_1}\cdot\ldots\cdot\mathbb{E}\,\xi_n^{q_n} \\
&= \sum_{\ell=1}^q \sum_{1\le i_1<i_2<\ldots<i_\ell\le n} \sum_{\substack{q_{i_1}+\ldots+q_{i_\ell}=q \\ q_{i_1}\ge 1,\ldots,q_{i_\ell}\ge 1}} \binom{q}{q_{i_1},\ldots,q_{i_\ell}} \tau^\ell \\
&\le \sum_{\ell=1}^q \frac{n^\ell}{\ell!}\ell^q\tau^\ell \le \sum_{\ell=1}^q \binom{q}{\ell} q^{q-\ell}\frac{\ell^q}{q!}(n\tau)^\ell \le \sum_{\ell=1}^q \binom{q}{\ell} q^{q-\ell}(e\tau n)^\ell \\
&\le (q+e\tau n)^q.
\end{aligned} \tag{4.25}$$

It is perhaps more natural (and also more precise) to estimate q^{th} moments by means of the Bernoulli law

$$\mathbb{E}\Big[\sum_{j=1}^n \xi_j\Big]^q = \sum_{\ell=0}^n \binom{n}{\ell} \ell^q \tau^\ell (1-\tau)^{n-\ell}.$$

But we have found the approach leading to (4.25) more flexible since it also applies to non Bernoulli cases. Continuing with the final term in (4.24) one concludes from (4.25) that

$$\sup_{|y|\le 1}\Big\|\sum_{\rho_1<y_i<\rho_2}\xi_i(\omega)\,y_i\Big\|_{L^q(\omega)} \le 2\sum_{\rho_2^{-2}<2^j<\rho_1^{-2}} 2^{-\frac{j}{2}}\Big\|\sum_{i=1}^{2^j}\xi_i(\omega)\Big\|_{L^q(\omega)} \tag{4.26}$$

$$\le 2\sum_{\rho_2^{-2}<2^j<\rho_1^{-2}} 2^{-\frac{j}{2}}(q+e\tau 2^j) \lesssim q\rho_2+\tau\rho_1^{-1}. \tag{4.27}$$

Inserting this bound into (4.24) and setting $\rho_1=\sqrt{\tau/m}$ and $\rho_2=q^{-\frac12}$ yields

$$\mathbb{E}\sup_{y\in\mathcal{F},\,|A|\le m}\sum_{i\in A}\xi_i y_i \lesssim \sqrt{m\tau}+\sqrt{q}\lesssim\sqrt{m\tau}+1+\sqrt{\log\#\mathcal{F}}.$$

The lemma now follows in view of (4.19) and (4.20). □

4.4 Entropy bounds

As in [B1] we will need bounds on certain covering numbers, also called entropies. We recall those bounds starting with the so called "dual Sudakov inequality" for the reader's convenience. More on this can be found in Pisier [P] and Bourgain, Lindenstrauss, Milman [BLM], Section 4. Consider $\mathbb{R}^n$ with two norms, the Euclidean norm $|\cdot|$ and some other (semi)norm $\|\cdot\|$. We set $X=(\mathbb{R}^n,\|\cdot\|)$ and denote the unit ball in this space by B_X, whereas the Euclidean unit ball will be B^n. As usual, for any set $U\subset\mathbb{R}^n$ and $t>0$ one sets

$$E(U,B_X,t):=\inf\Big\{N\ge 1\mid \exists\, x_j\in\mathbb{R}^n,\ 1\le j\le N,\ U\subset\bigcup_{j=1}^N(x_j+tB_X)\Big\}. \tag{4.28}$$

There are two closely related quantities, namely

$$\tilde E(U,B_X,t) := \inf\Big\{N\ge 1\mid \exists\, x_j\in U,\ 1\le j\le N,\ U\subset\bigcup_{j=1}^N(x_j+tB_X)\Big\}$$

$$D(U,B_X,t) := \sup\Big\{M\ge 1\mid \exists\, y_j\in U,\ 1\le j\le M,\ \|y_j-y_k\|\ge t,\ j\ne k\Big\}. \tag{4.29}$$

There are the following comparisons between these quantities:

$$D(U,B_X,t)\ge \tilde E(U,B_X,t)\ge E(U,B_X,t)\ge D(U,B_X,2t) \tag{4.30}$$

The final inequality holds because every covering of U by arbitrary t-balls gives rise to a covering by $2t$-balls with centers in U. To see that $E(U,B_X,t)\ge D(U,B_X,2t)$, let $\{y_j\}_{j=1}^M\subset U$ be $2t$-separated and $U\subset\bigcup_{i=1}^N(x_i+tB_X)$. Then every $y_j\in x_i+tB_X$ for some $i=i(j)$. Moreover, $j\ne k \implies i(j)\ne i(k)$. Hence $N\ge M$.
The "dual Sudakov inequality" Lemma 4.8 bounds $E(B^n,B_X,t)$ in terms of the Levy mean

$$M_X:=\int_{S^{n-1}}\|x\|\,d\sigma(x), \tag{4.31}$$

where σ is the normalized measure on S^{n-1}. Alternatively, one has

$$M_X = \alpha_n (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} \|x\| \, dx \tag{4.32}$$

$$M_X = \alpha_n \int_{\Omega} \Big\| \sum_{i=1}^{n} g_i(\omega) \vec{e_i} \Big\| \, d\mathbb{P}(\omega), \tag{4.33}$$

where

$$\alpha_n = \frac{\Gamma\Big(\frac{n}{2}\Big)}{\Gamma\Big(\frac{n+1}{2}\Big)\sqrt{2}} \asymp n^{-\frac{1}{2}}$$

and g_i are i.i.d. standard normal variables, and $\vec{e_i}$ is an ONS. The probabilistic form (4.33) is of course just a restatement of (4.32), whereas the latter can be obtained from the definition (4.31) by means of polar coordinates. The following lemma is due to Pajor and Tomczak-Jaegerman [PT-J] but the proof given below is due to Pajor and Talagrand, see [BLM].

Lemma 4.8. *For any $t > 0$*

$$\log E(B^n, B_X, t) \le Cn \Big(\frac{M_X}{t} \Big)^2, \tag{4.34}$$

where C is an absolute constant.

Proof. Let $\{x_i\}_{i=1}^N \subset B^n$, $\|x_i - x_j\| \ge t$ for $i \ne j$ and N maximal. Then $E(B^n, B_X, t) \le N$. Let $\mu(dx) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \, dx$. Then by definition (4.31),

$$\mu(\|x\| > 2M_X\alpha_n^{-1}) < \frac{1}{2} \implies \mu(\|x\| \le 2M_X\alpha_n^{-1}) > \frac{1}{2}. \tag{4.35}$$

Moreover, $\{x_i + \frac{1}{2}tB_X\}_{i=1}^N$ and therefore also $\{y_i + 2M_X\alpha_n^{-1}B_X\}_{i=1}^N$ have mutually disjoint interiors, where we have set $y_i = 4M_X(t\alpha_n)^{-1}x_i$. Now, by symmetry of B_X and convexity of e^{-u},

$$\begin{aligned}
\mu(y_i + 2M_X\alpha_n^{-1}B_X) &= (2\pi)^{-\frac{n}{2}} \int_{2M_X\alpha_n^{-1}B_X} e^{-|y-y_i|^2/2} \, dy \\
&= (2\pi)^{-\frac{n}{2}} \int_{2M_X\alpha_n^{-1}B_X} \frac{1}{2} \Big[e^{-|y-y_i|^2/2} + e^{-|y+y_i|^2/2} \Big] \, dy \\
&\ge (2\pi)^{-\frac{n}{2}} \int_{2M_X\alpha_n^{-1}B_X} e^{-(|y-y_i|^2+|y+y_i|^2)/4} \, dy \\
&= (2\pi)^{-\frac{n}{2}} \int_{2M_X\alpha_n^{-1}B_X} e^{-(|y|^2+|y_i|^2)/2} \, dy \ge \frac{1}{2} e^{-|y_i|^2/2},
\end{aligned}$$

where the last step follows from (4.35). Since $|y_i| \le 4M_X(t\alpha_n)^{-1}$,

$$\mu(y_i + 2M_X\alpha_n^{-1}B_X) \ge \frac{1}{2} \exp\Big(-\frac{1}{2}(4M_X)^2(t\alpha_n)^{-2} \Big).$$

Hence

$$1 \geq \sum_{i=1}^{N} \mu(y_i + 2M_X \alpha_n^{-1} B_X) \geq \frac{1}{2} N \exp\Big(- (4M_X)^2 (t\alpha_n)^{-2} \Big),$$

and the lemma follows since $\alpha_n \asymp n^{-\frac{1}{2}}$. $\square$

Observe that (4.34) is a poor bound as $t \to 0$. Indeed, rather than the $\exp(t^{-2})$ behavior exhibited by (4.34) the true asymptotics is t^{-n} as $t \to 0$. The point of Lemma 4.8 is to relate the size of t to both M_X and n. This is best illustrated by some standard examples.

- Firstly, take $X = \ell_n^1$. In that case,

$$\alpha_n^{-1} M_X = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^{n} |x_i| e^{-\frac{|x|^2}{2}} \, dx = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x_1| e^{-\frac{x_1^2}{2}} \, dx_1 = \frac{2n}{\sqrt{2\pi}}.$$

  Therefore, $M_X \asymp \sqrt{n}$. By (4.34),

$$\sup_n E(B^n, B_{\ell_n^1}, n) \leq C.$$

  This bound is somewhat wasteful. Indeed, since $\sqrt{n} B_{\ell_n^1} \supset B^n$, one actually has

$$\sup_n E(B^n, B_{\ell_n^1}, \sqrt{n}) \leq C.$$

  The reason for this "overshoot" is that the major contribution to M_X comes from the corners of $B_{\ell_n^1}$. On the other hand, these corners do not determine the smallest r for which $rB_X \supset B^n$.

- Secondly, consider $X = \ell_n^\infty$. Using (4.33),

$$\alpha_n^{-1} M_X = \mathbb{E} \sup_{1 \leq i \leq n} |g_i| \asymp \sqrt{\log n},$$

  where the latter bound is a rather obvious and well-known fact. Hence

$$M_X \asymp \sqrt{\frac{\log n}{n}}$$

  which implies via (4.34) that

$$\sup_n E(B^n, B_{\ell_n^\infty}, \sqrt{\log n}) \leq C.$$

  This is the correct behavior up to the $\log n$-factor since $B^n \subset B_{\ell_n^\infty}$. In contrast to the previous case, the bulk of the contribution to M_X comes from that part of $B_{\ell_n^\infty}$ that is also the most relevant for the covering of the Euclidean ball.

- Finally, and most relevantly for our purposes, identify $\mathbb{R}^n$ with the space of trigonometric polynomials with real coefficients of degree n, i.e.,

$$\mathbb{R}^n \simeq \Big\{ \sum_{j=1}^{n} a_j e(j\theta) \mid a_j \in \mathbb{R} \Big\}. \tag{4.36}$$

Furthermore, define $\| \cdot \| = \| \cdot \|_{L^q(\mathbb{T})}$ where $q \geq 2$ is fixed. Then

$$\begin{aligned} M_X &= \alpha_n \int_\Omega \Big\| \sum_{j=1}^{n} g_j(\omega) e(j\theta) \Big\|_{L^q(\mathbb{T})} d\mathbb{P}(\omega) \\ &= \alpha_n \, \mathbb{E} \int_\Omega \Big\| \sum_{j=1}^{n} \pm g_j(\omega) e(j\theta) \Big\|_{L^q(\mathbb{T})} d\mathbb{P}(\omega) \qquad (4.37) \\ &\leq C \alpha_n \sqrt{q} \int_\Omega \Big(\sum_{j=1}^{n} g_j^2(\omega) \Big)^{\frac{1}{2}} d\mathbb{P}(\omega) \qquad (4.38) \\ &\leq C \alpha_n \sqrt{q} \left(\int_\Omega \sum_{j=1}^{n} g_j^2(\omega) \, d\mathbb{P}(\omega) \right)^{\frac{1}{2}} = C \alpha_n \sqrt{q} \sqrt{n} \leq C \sqrt{q}. \end{aligned}$$

In (4.37) the expectation $\mathbb{E}$ refers to the random and symmetric choice of signs $\pm$, whereas the $\sqrt{q}$-factor in (4.38) is due to the fact that the constant in Khinchin's inequality grows like $\sqrt{q}$. Hence

$$\log E(B^n, B_X, t) \leq C\, qnt^{-2} \tag{4.39}$$

in this case.

The proof of Proposition 4.6 requires estimating $N_q(\mathcal{P}_A, t) := E(\mathcal{P}_A, B_{L^q(\mathbb{T})}, t)$. Here

$$\mathcal{P}_A := \Big\{ \sum_{n \in A} a_n e(n\theta) \;\Big|\; |a| = |a|_{\ell^2_N} \leq 1 \Big\}$$

where $A \subset [1, N]$. Invoking (4.39) leads to

$$\log N_q(\mathcal{P}_A, t) \leq Cq|A|\, t^{-2}. \tag{4.40}$$

This bound is basically optimal when $t \sim 1$, but it can be improved for very small and very large t.

Corollary 4.9. *For $q \geq 2$ and any $A \subset [1, N]$*

$$\log N_q(\mathcal{P}_A, t) \leq C\, q|A| \Big[1 + \log \frac{1}{t} \Big] \quad \text{if} \;\; 0 < t \leq \frac{1}{2}. \tag{4.41}$$

Proof. Let $m = |A|$. Thus $1 \le m \le N$. Notice firstly that

$$\log N_q\Big(\Big\{\sum_{n\in A} a_n e(n\theta)\,\Big|\,|a|\le 1\Big\}, t\Big) \le C\,m\log\frac{1}{t} + \log N_q\Big(\Big\{\sum_{n\in A} a_n e(n\theta)\,\Big|\,|a|\le 1\Big\}, 1\Big). \tag{4.42}$$

This follows from the fact that for any norm $\|\cdot\|$ in $\mathbb{R}^m$ with unit-balls B_X one has

$$D(B_X, B_X, t) \le (4/t)^m \quad \text{for all} \quad 0 < t < 1 \tag{4.43}$$

by scaling and volume counting, see (4.29) for the definition of $D(B_X, B_X, t)$. Indeed, suppose $M = D(B_X, B_X, t)$. Then there are M disjoint balls $\{x_j + \frac{1}{2}tB_X\}_{j=1}^M$ with centers $x_j \in B_X$. Since $x_j + \frac{1}{2}tB_X \subset 2B_X$ if $t < 1$, it follows that

$$\sum_{j=1}^M |\frac{1}{2}tB_X| \le |2B_X| \implies M(t/2)^m \le 2^m,$$

as claimed. Here $|\cdot|$ stands for Lebesgue measure. Thus (4.43) holds, and therefore also (4.42) in view of (4.30). Hence

$$\begin{aligned}\log N_q(\mathcal{P}_A, t) &\le C\,m\log\frac{1}{t} + \log N_q\Big(\Big\{\sum_{n\in A} a_n e(n\theta)\,\Big|\,|a|\le 1\Big\}, 1\Big)\\ &\le C\,m\log\frac{1}{t} + Cqm,\end{aligned}$$

where the final term follows from (4.39). □

We now turn to large t. The following corollary slightly improves on the rate of decay.

Corollary 4.10. *Let $q \ge 2$ and $A \subset [1, N]$. With $\mathcal{P}_A$ as above one has*

$$\log N_q(\mathcal{P}_A, t) \le Cq\,|A|\,t^{-\nu} \quad \textit{if} \quad t > \frac{1}{2} \tag{4.44}$$

where $\nu = \nu(q) > 2$.

Proof. Recall that $N_q(\mathcal{P}_A, t) = E(\mathcal{P}_A, B_{L^q}, t)$. Using (4.30), one obtains from (4.40) that also

$$\log \tilde{E}(\mathcal{P}_A, B_{L^q}, t) \le Cq\,|A|\,t^{-2}. \tag{4.45}$$

Let $q < r$, $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$. Since for any $f, g \in \mathcal{P}_A$

$$\|f-g\|_q \le \|f-g\|_2^{1-\theta}\|f-g\|_r^{\theta} \le 2\|f-g\|_r^{\theta},$$

one concludes from (4.45) that

$$\log \tilde{E}(\mathcal{P}_A, B_{L^q}, t) \le \log \tilde{E}(\mathcal{P}_A, B_{L^r}, (t/2)^{1/\theta}) \le Cq\,|A|\,t^{-2/\theta}.$$

Applying (4.30) again yields (4.44). □

4.5 Decoupling lemma

Lastly, we require a version of Bourgain's decoupling technique, cf. Lemma 4 in [B1]. In contrast to his case we only need to decouple into two sets rather than three.

Lemma 4.11. *Let real-valued functions $h_\alpha(u)$ on $\mathbb{C}$ be given for $\alpha = 1,2,3$ that satisfy*

$$|h_\alpha(u)| \le (1+|u|)^{p_\alpha}, \quad |h_\alpha(u) - h_\alpha(v)| \le (1+|u|+|v|)^{p_\alpha-\delta}\,|u-v|^\delta$$

for all $u, v \in \mathbb{C}$ and some fixed choice of $p_\alpha > 0$, $\delta > 0$. Let $x, y, z \in \ell^2_N$ be sequences so that $|x|, |y|, |z| \le 1$ and suppose $\zeta_j = \zeta_j(t)$ are i.i.d. random variables with $\mathbb{P}(\zeta_j = 1) = \mathbb{P}(\zeta_j = 0) = \frac{1}{2}$. We assume that $\mathbb{P}(dt) = dt$ on $[0,1]$, say. Set $R^1_t = \{1 \le j \le N | \zeta_j(t) = 1\}$, $\quad R^2_t = \{1 \le j \le N | \zeta_j(t) = 0\}$. Then

$$\begin{aligned}&\Big|\int h_1\Big(\sum_{i\in R^1_t} x_i\Big) h_2\Big(\sum_{i\in R^2_t} y_i\Big) h_3\Big(\sum_{i\in R^2_t} z_i\Big)\, dt - h_1\Big(\frac{1}{2}\sum_i x_i\Big) h_2\Big(\frac{1}{2}\sum_i y_i\Big) h_3\Big(\frac{1}{2}\sum_i z_i\Big)\Big| \\ &\le C\,\Big(1 + \Big|\sum_i x_i\Big| + \Big|\sum_i y_i\Big| + \Big|\sum_i z_i\Big|\Big)^{p-\delta} \qquad (4.46)\end{aligned}$$

where $p = p_1 + p_2 + p_3$ and C is some absolute constant depending only on p and δ.

Proof. By assumption,

$$\begin{aligned}\Big|h_\alpha\Big(\sum_{i\in R^1_t} x_i\Big) - h_\alpha\Big(\frac{1}{2}\sum_{i=1}^N x_i\Big)\Big| &\le \Big(1 + |\sum_{i=1}^N x_i| + |\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i|\Big)^{p_\alpha-\delta} \Big|\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i\Big|^\delta \\ &\le \Big(1 + |\sum_{i=1}^N x_i|\Big)^{p_\alpha-\delta}\Big(1 + |\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i|\Big)^{p_\alpha} \\ \Big|h_\alpha\Big(\sum_{i\in R^1_t} x_i\Big)\Big| + \Big|h_\alpha\Big(\frac{1}{2}\sum_{i=1}^N x_i\Big)\Big| &\le 2\Big(1 + |\sum_{i=1}^N x_i|\Big)^{p_\alpha}\Big(1 + |\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i|\Big)^{p_\alpha}\end{aligned}$$

for $\alpha = 1,2,3$. Hence

$$\begin{aligned}&\Big|\int h_1\Big(\sum_{i\in R^1_t} x_i\Big) h_2\Big(\sum_{i\in R^2_t} y_i\Big) h_3\Big(\sum_{i\in R^2_t} z_i\Big)\, dt - h_1\Big(\frac{1}{2}\sum_{i=1}^N x_i\Big) h_2\Big(\frac{1}{2}\sum_{i=1}^N y_i\Big) h_3\Big(\frac{1}{2}\sum_{i=1}^N z_i\Big)\Big| \\ &\le C\Big(1 + \Big|\sum_{i=1}^N x_i\Big| + \Big|\sum_{i=1}^N y_i\Big| + \Big|\sum_{i=1}^N z_i\Big|\Big)^{p-\delta} \\ &\qquad \cdot \int \Big(1 + \Big|\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i\Big| + \Big|\sum_{i=1}^N (\zeta_i - \frac{1}{2}) y_i\Big| + \Big|\sum_{i=1}^N (\zeta_i - \frac{1}{2}) z_i\Big|\Big)^p\, dt. \qquad (4.47)\end{aligned}$$

The lemma now follows from Khinchin's inequality. Indeed,

$$\int \Big|\sum_{i=1}^N (\zeta_i - \frac{1}{2}) x_i\Big|^p\, dt \le C_p\, |x|^p \le C_p,$$

by assumption. □

4.6 The proof of Proposition 4.6 and its improvement for $p = 3$

We now start the proof of Proposition 4.6 for $p = 3$. In fact, we state a somewhat more precise form of this theorem for $p = 3$.

Theorem 4.12. *Let $0 < \delta < 1$ be fixed. For every positive integer N we let $\xi_j = \xi_j(\omega)$ be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$ where $\tau = N^{-\delta}$. Define a random subset*

$$S(\omega) = \{j \in [1, N] \mid \xi_j(\omega) = 1\}.$$

Then for every $\gamma > 0$ there is a constant C_γ so that

$$\sup_{N\geq 1} \mathbb{P}\Big[\sup_{|a_n|\leq 1} \Big\| \sum_{n\in S(\omega)} a_n e(n\theta)\Big\|_{L^3(\mathbb{T})} \geq C_\gamma \Big\| \sum_{n\in S(\omega)} e(n\theta)\Big\|_{L^3(\mathbb{T})}\Big] \leq \gamma. \tag{4.48}$$

Proof. Firstly, note that for fixed $0 < \delta < 1$ and large N Lemma 4.1 implies that

$$\mathbb{P}\Big[\sum_{n=1}^N \xi_n \geq 2\tau N\Big] \lesssim \exp(-c\tau N).$$

Let $\mathbb{E}'$ denote the restricted expectation

$$\mathbb{E}' \sup_{|a_n|\leq 1} \Big\|\sum_{n=1}^N \xi_n a_n e(n\theta)\Big\|_{L^3(\mathbb{T})} := \mathbb{E}\,\chi_{[\sum \xi_n \leq 2\tau N]} \sup_{|a_n|\leq 1} \Big\|\sum_{n=1}^N \xi_n a_n e(n\theta)\Big\|_{L^3(\mathbb{T})}.$$

Then

$$\begin{aligned}\mathbb{E} \sup_{|a_n|\leq 1} \Big\|\sum_{n=1}^N \xi_n a_n e(n\theta)\Big\|_{L^3(\mathbb{T})} &\leq N\exp(-c\tau N) + \mathbb{E}' \sup_{|a_n|\leq 1} \Big\|\sum_{n=1}^N \xi_n a_n e(n\theta)\Big\|_{L^3(\mathbb{T})} \\ &\leq O(1) + \mathbb{E}' \sup_{|a_n|\leq 1} \Big\|\sum_{n=1}^N \xi_n a_n e(n\theta)\Big\|_{L^3(\mathbb{T})}.\end{aligned}$$

From now on, we set $m = 2\tau N$, and we will mostly work with $\mathbb{E}'$ instead of $\mathbb{E}$. Next, fix some $\{a_n\}_{n=1}^N$ with $|a_n| \leq 1$. Then, rescaling Lemma 4.11 (with $h_1(x) = h_2(x) = x$ and $h_3(x) = |x|$) one obtains that

$$\begin{aligned}\frac{1}{8}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n e(n\theta)\Big|^3 d\theta &= \int\!\!\int_0^1 \sum_{n\in R_t^1} a_n\xi_n e(n\theta) \sum_{k\in R_t^2} \bar{a}_k\xi_k e(-k\theta)\Big|\sum_{\ell\in R_t^2} a_\ell\xi_\ell e(\ell\theta)\Big|\,d\theta\,dt \\ &\quad + O\left(m^{\frac{3}{2}}\int_0^1 \Big(1 + \Big|\sum_{n=1}^N \frac{a_n}{\sqrt{m}}\xi_n e(n\theta)\Big|^2\Big)\,d\theta\right).\end{aligned} \tag{4.49}$$

The O-term in (4.49) is $O(m^{\frac{3}{2}})$ by construction. Let $\{\xi_n(\omega_1)\}_{n=1}^N$ and $\{\xi_n(\omega_2)\}_{n=1}^N$ denote two independent copies of $\{\xi_n(\omega)\}_{n=1}^N$. Recall that R_t^1 and R_t^2 are disjoint for every t. Therefore, *for* *fixed t*

$$\mathbb{E}_{\omega}\sup_{|a_n|\le 1}\left|\int_0^1 \sum_{n\in R_t^1} a_n\xi_n(\omega)e(n\theta)\sum_{k\in R_t^2}\bar a_k\xi_k(\omega)e(-k\theta)\Big|\sum_{\ell\in R_t^2} a_\ell\xi_\ell(\omega)e(\ell\theta)\Big|\,d\theta\right|$$

$$= \mathbb{E}_{\omega_1,\omega_2}\sup_{|a_n|\le 1}\left|\int_0^1 \sum_{n\in R_t^1} a_n\xi_n(\omega_1)e(n\theta)\sum_{k\in R_t^2}\bar a_k\xi_k(\omega_2)e(-k\theta)\Big|\sum_{\ell\in R_t^2} a_\ell\xi_\ell(\omega_2)e(\ell\theta)\Big|\,d\theta\right|. \tag{4.50}$$

This leads to

$$\begin{aligned}
&\mathbb{E}_{\omega}\sup_{|a_n|\le 1}\int_0^1\Big|\sum_{n=1}^N a_n\xi_n(\omega)e(n\theta)\Big|^3\,d\theta\\
\lesssim\ & m^{\frac32}+\int \mathbb{E}_{\omega_1,\omega_2}\sup_{|a_n|\le 1}\left|\int_0^1 \sum_{n\in R_t^1} a_n\xi_n(\omega_1)e(n\theta)\sum_{k\in R_t^2}\bar a_k\xi_k(\omega_2)e(-k\theta)\Big|\sum_{\ell\in R_t^2} a_\ell\xi_\ell(\omega_2)e(\ell\theta)\Big|\,d\theta\right|dt\\
\lesssim\ & m^{\frac32}+\int \mathbb{E}'_{\omega_1}\mathbb{E}'_{\omega_2}\sup_{\substack{|a_n|\le 1\\|b_n|\le 1}}\left|\int_0^1 \sum_{n\in R_t^1} a_n\xi_n(\omega_1)e(n\theta)\sum_{k\in R_t^2}\bar b_k\xi_k(\omega_2)e(-k\theta)\Big|\sum_{\ell\in R_t^2} b_\ell\xi_\ell(\omega_2)e(\ell\theta)\Big|\,d\theta\right|dt\\
\lesssim\ & m^{\frac32}+\int \mathbb{E}'_{\omega_1}\mathbb{E}'_{\omega_2}\sup_{\substack{|a_n|\le 1\\|b_n|\le 1}}\left|\int_0^1 \sum_{n=1}^N a_n\xi_n(\omega_1)e(n\theta)\sum_{k=1}^N\bar b_k\xi_k(\omega_2)e(-k\theta)\Big|\sum_{\ell=1}^N b_\ell\xi_\ell(\omega_2)e(\ell\theta)\Big|\,d\theta\right|dt\\
\lesssim\ & m^{\frac32}+\mathbb{E}'_{\omega_2}\mathbb{E}_{\omega_1}\sup_{x\in\mathcal{E}(\omega_2)}\sup_{|A|=m}\sum_{n\in A}\xi_n(\omega_1)x_n.
\end{aligned}\tag{4.51}$$

Here

$$\mathcal{E}(\omega_2):=\left\{\left(\Big|\Big\langle e(n\cdot),\sum_{k=1}^N\bar b_k\xi_k(\omega_2)e(-k\cdot)\Big|\sum_{\ell=1}^N b_\ell\xi_\ell(\omega_2)e(\ell\cdot)\Big|\Big\rangle\Big|\right)_{n=1}^N\ \Big|\ \sup_{1\le n\le N}|b_n|\le 1\right\}\subset\mathbb{R}_+^N.$$

In the calculation leading up to (4.51) we firstly used (4.50), secondly the obvious fact that the supremum only increases if we introduce $\{b_n\}_{n=1}^N$ in addition to $\{a_n\}_{n=1}^N$, thirdly that one can remove the restrictions to the sets R_t^1 and R_t^2 because they can be absorbed into the choice of the sequences a_n, b_n, and lastly that $\sum_n \xi_n\le m$ which allows us to introduce $A\subset[1,N]$, $|A|=m$. If $x\in\mathcal{E}(\omega_2)$, then

$$|x|^2_{\ell^2_N}\le\sup_{|a_k|\le 1}\Big\|\sum_k a_k\,\xi_k(\omega_2)e(k\cdot)\Big\|_4^4\le\Big\|\sum_k\xi_k(\omega_2)e(k\cdot)\Big\|_4^4=:B_4^2(\omega_2) \tag{4.52}$$

by the L^4 majorant property. By Lemma 4.3,

$$\mathbb{E}\,B_4\le\Big(\mathbb{E}\,I_{4,N}\Big)^{\frac12}\lesssim\tau^2N^{\frac32}+\tau N. \tag{4.53}$$

We now apply Lemma 4.7 to (4.51). This yields

$$\mathbb{E}_{\omega} \sup_{|a_n|\le 1} \int_0^1 \Big|\sum_{n=1}^N a_n \xi_n(\omega) e(n\theta)\Big|^3 \, d\theta \lesssim m^{\frac{3}{2}} + \mathbb{E}'_{\omega_2}\left[(\sqrt{\tau m}+1)\, B_4(\omega_2) + \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt \right]$$

$$\lesssim (\tau N)^{\frac{3}{2}} + (1+\tau N^{\frac{1}{2}})(\tau^2 N^{\frac{3}{2}} + \tau N) + \mathbb{E}'_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt. \tag{4.54}$$

It remains to deal with the entropy integral in (4.54). To this end, observe that the distance between any two elements in $\mathcal{E}(\omega_2)$ is of the form

$$\begin{aligned} \|g|g| - h|h|\|_2 &\le \|g-h\|_\infty (\|g\|_2 + \|h\|_2) \\ &\lesssim N^\varepsilon \|g-h\|_q (\|g\|_2 + \|h\|_2) \lesssim N^\varepsilon \sqrt{m} \|g-h\|_q, \end{aligned}$$

where we chose q very large depending on ε (the factor N^ε comes from Bernstein's inequality). Here $g, h \in \sqrt{m}\mathcal{P}_A$ where $A = A(\omega_2) = \{n \in [1,N] \mid \xi_n(\omega_2) = 1\}$ and

$$\mathcal{P}_A = \left\{ \sum_{n\in A} a_n e(n\cdot) \,\Big|\, |a|_{\ell^2_N} \le 1 \right\}. \tag{4.55}$$

Actually, our coefficients are in the unit-ball of ℓ^∞_n, but we have embedded this into ℓ^2_m in the obvious way, which leads to the $\sqrt{m}$-factor in front of $\mathcal{P}_A$ (at this point recall that we are working with $\mathbb{E}'_{\omega_2}$). One concludes that, for $\varepsilon > 0$ small and $q < \infty$ large depending on ε,

$$\begin{aligned} \log N_2(\mathcal{E}(\omega_2),t) &\le \log N_q(\mathcal{P}_A, N^{-\varepsilon} m^{-1} t) \\ &\le Cq\, m \begin{cases} 1 + \log \frac{mN^\varepsilon}{t} & 0 < t < mN^\varepsilon \\ (m^{-1}N^{-\varepsilon} t)^{-\nu} & t > N^\varepsilon m \end{cases} \end{aligned} \tag{4.56}$$

where $\nu > 2$, see Corollary 4.9 and Corollary 4.10. It follows that the last term in (4.54) is at most

$$\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt \lesssim N^\varepsilon m^{\frac{3}{2}}.$$

Plugging this into (4.54) yields

$$\begin{aligned} \mathbb{E}_{\omega} \sup_{|a_n|\le 1} \int_0^1 \Big|\sum_{n=1}^N a_n \xi_n(\omega) e(n\theta)\Big|^3 \, d\theta &\lesssim (\tau N)^{\frac{3}{2}} + (1+\tau N^{\frac{1}{2}})(\tau^2 N^{\frac{3}{2}} + \tau N) + N^\varepsilon (\tau N)^{\frac{3}{2}} \\ &\lesssim \tau^3 N^2 + N^\varepsilon (\tau N)^{\frac{3}{2}}. \end{aligned} \tag{4.57}$$

Now suppose $\delta < \frac{1}{3}$. Then $\tau^3 N^2 > N^\varepsilon (\tau N)^{\frac{3}{2}}$ provided $\varepsilon > 0$ is small and fixed, and provided N is large. Hence, combining (4.57) with Lemma 4.3 leads to Theorem 4.12 at least if $\delta < \frac{1}{3}$. If one is willing to loose a N^ε-factor, then (4.57) in combination with Lemma 4.3 leads to the desired bounds in all cases. On the other hand, if $\delta \ge \frac{1}{3}$ so that typically $\#(S(\omega)) \lesssim N^{\frac{2}{3}}$, then Bourgain showed that $S(\omega)$ is a Λ_3 set with large probability. More precisely, he showed that the constant

$$K_3(\omega) := \sup_{|a|_{\ell^2_N} \le 1} \Big\| \sum_{n\in S(\omega)} a_n e(n\cdot) \Big\|_3$$

satisfies $\mathbb{E}\, K_3^3 \le C$. Hence, in our case,

$$\mathbb{E} \sup_{|a_n|\le 1} \Big\| \sum_{n=1}^{N} a_n \xi_n(\omega) e(n\cdot) \Big\|_3^3 \lesssim (\tau N)^{\frac{3}{2}}.$$

Clearly,

$$\Big\| \sum_{n=1}^{N} \xi_n(\omega) e(n\cdot) \Big\|_3 \ge \Big\| \sum_{n=1}^{N} \xi_n(\omega) e(n\cdot) \Big\|_2 = \#(S(\omega))^{\frac{1}{2}},$$

and we have thus proved (4.48) for $\delta \ge \frac{1}{3}$ as well. $\square$

It is perhaps worth pointing out that interpolation of the L^4 bound with the L^2 bound gives

$$\tau^{\frac{5}{2}} N^2 + (\tau N)^{\frac{3}{2}},$$

so that the estimate we just obtained is better by the initial τ^3-factor (note that this is due to the $\sqrt{\tau m}$-factor in Lemma 4.7 as compared to a $\sqrt{\tau N}$-factor).

4.7 The case of general p

The strategy is to first generalize the previous argument to all odd integers using the fact that the majorant property holds for all even integers (for $p = 3$ we used this fact with $p = 4$). Then one runs the same argument again, using now that the (random) majorant property holds for *all* integers p and so on. For a given $\varepsilon > 0$ this yields that there is a set of p that is ε-dense in $[2, \infty)$ and for which the majorant property holds. This is enough by interpolation, since we are allowing a loss of N^ε in (4.18). Unfortunately, there are certain technical complications in carrying out this program having to do with the size of δ. In this section we finish a proof a Proposition 4.6 by employing the above method for $p = 3$. The next Lemma formalizes the main probabilistic argument from the previous section. Let $p \ge 2$. In this section, we say that the *random majorant property* (or RMP in short) holds at p if and only if for every $\varepsilon > 0$ there exists a constant C_ε so that

$$\mathbb{E} \sup_{|a_n|\le 1} \Big\| \sum_{n=1}^{N} a_n \xi_n e(n\theta) \Big\|_p^p \le C_\varepsilon\, N^\varepsilon\, \mathbb{E} \Big\| \sum_{n=1}^{N} \xi_n e(n\theta) \Big\|_p^p \tag{4.58}$$

for all $N \ge 1$. Note that (the proof of) Theorem 4.12 establishes that the random majorant property holds at $p = 3$. Moreover, if (4.58) holds for some p, then (4.18) also holds for that value of p, see Lemma 4.3.

Lemma 4.13. *Let $2 \le p \le 3$. Suppose the random majorant property (4.58) holds at $2(p-1)$. Then it also holds at p. Furthermore, suppose the RMP holds at $p-1$, $2(p-1)$ and $2(p-2)$. If $4 \ge p \ge 3$, then it also holds at p. If $p > 4$ and $\delta \le \frac{1}{2}$ (i.e., $\tau = N^{-\delta} \ge N^{-\frac{1}{2}}$), then it also holds at p.*

Proof. Assume first that $p \geq 3$. Instead of (4.49), Lemma 4.11 implies in this case that

$$
\begin{aligned}
2^{-p}\int_0^1 \left|\sum_{n=1}^N a_n\xi_n e(n\theta)\right|^p d\theta \quad &= \quad \int\int_0^1 \sum_{n\in R_t^1} a_n\xi_n e(n\theta)\sum_{k\in R_t^2} \bar a_k\xi_k e(-k\theta)\Big|\sum_{\ell\in R_t^2} a_\ell\xi_\ell e(\ell\theta)\Big|^{p-2} d\theta\, dt \\
&\quad + O\left(m^{\frac{p}{2}}\int_0^1 \Big(1+\Big|\sum_{n=1}^N \frac{a_n}{\sqrt{m}}\xi_n e(n\theta)\Big|^{p-1}\Big)\, d\theta\right). \qquad (4.59)
\end{aligned}
$$

To bound the O-term in (4.59) note that by the RMP for $p-1\geq 2$,

$$
\mathbb{E}\sup_{|a_n|\leq 1}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n e(n\theta)\Big|^{p-1} d\theta \leq C_\varepsilon\, N^\varepsilon\, \mathbb{E}\int_0^1 \Big|\sum_{n=1}^N \xi_n e(n\theta)\Big|^{p-1} d\theta = C_\varepsilon\, N^\varepsilon\, \mathbb{E}\, I_{p-1,N}. \qquad (4.60)
$$

A calculation analogous to that leading up to (4.51) therefore yields

(4.61)

$$
\mathbb{E}_\omega \sup_{|a_n|\leq 1}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n(\omega)e(n\theta)\Big|^p d\theta \lesssim m^{\frac{p}{2}} + C_\varepsilon\, N^\varepsilon m^{\frac12}\,\mathbb{E}\, I_{p-1,N} + \mathbb{E}'_{\omega_2}\,\mathbb{E}'_{\omega_1}\sup_{x\in\mathcal{E}(\omega_2)}\ \sup_{|A|=m}\sum_{n\in A}\xi_n(\omega_1)x_n,
$$

where now

$$
\mathcal{E}(\omega_2) = \Big\{\Big(\Big|\Big\langle e(n\cdot), \sum_{k=1}^N \bar b_k\xi_k(\omega_2)e(-k\cdot)\Big|\sum_{\ell=1}^N b_\ell\xi_\ell(\omega_2)e(\ell\cdot)\Big|^{p-2}\Big\rangle\Big|\Big)_{n=1}^N \ \Big|\ \sup_{1\leq n\leq N}|b_n|\leq 1\Big\} \subset \mathbb{R}_+^N.
$$

If $x\in\mathcal{E}(\omega_2)$, then by Plancherel and the RMP at $2(p-1)$,

$$
\begin{aligned}
\mathbb{E}\sup_{x\in\mathcal{E}(\omega_2)}|x|^2_{\ell^2_N} \quad &\leq \quad \mathbb{E}_{\omega_2}\sup_{|a_k|\leq 1}\int_0^1 \Big|\sum_k a_k\,\xi_k(\omega_2)e(k\theta)\Big|^{2(p-1)} d\theta \qquad (4.62) \\
&\leq \quad C_\varepsilon\, N^\varepsilon\,\mathbb{E}_{\omega_2}\int_0^1 \Big|\sum_k \xi_k(\omega_2)e(k\theta)\Big|^{2(p-1)} d\theta \leq C_\varepsilon\, N^\varepsilon\,\mathbb{E}\, I_{2(p-1),N}.
\end{aligned}
$$

Thus, by (4.61) and Lemma 4.7,

$$
\begin{aligned}
&\mathbb{E}_\omega\sup_{|a_n|\leq 1}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n(\omega)e(n\theta)\Big|^p d\theta \\
&\leq C_\varepsilon\, N^\varepsilon\Big[m^{\frac{p}{2}} + m^{\frac12}\,\mathbb{E}\, I_{p-1,N} + (1+\sqrt{m\tau})\sqrt{\mathbb{E}\, I_{2(p-1),N}} + \mathbb{E}'_{\omega_2}\int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt\Big]. \qquad (4.63)
\end{aligned}
$$

To estimate the entropy term, let q be very large depending on ε. Then the distance between any two elements in $\mathcal{E}(\omega_2)$ is of the form

$$
\begin{aligned}
\|g|g|^{p-2} - h|h|^{p-2}\|_2 \quad &\leq \quad \|g-h\|_\infty\big(\|g\|^{p-2}_{2(p-2)} + \|h\|^{p-2}_{2(p-2)}\big) \\
&\leq \quad C_\varepsilon\, N^\varepsilon\|g-h\|_q\big(\|g\|^{p-2}_{2(p-2)} + \|h\|^{p-2}_{2(p-2)}\big) \\
&\leq \quad C_\varepsilon\, N^\varepsilon\sup_{|a_n|\leq 1}\Big\|\sum_{n=1}^N a_n\xi_n(\omega_2)e(n\cdot)\Big\|^{p-2}_{2(p-2)}\ \|g-h\|_q, \\
&=: \quad C_\varepsilon\, N^\varepsilon J^{\frac12}_{2(p-2),N}(\omega_2)\ \|g-h\|_q, \qquad (4.64)
\end{aligned}
$$

where the N^ε-term follows from Bernstein's inequality and we have set

$$\sup_{|a_n|\le 1} \Big\| \sum_{n=1}^N a_n \xi_n(\omega_2) e(n\cdot) \Big\|_{2(p-2)}^{2(p-2)} =: J_{2(p-2),N}(\omega_2).$$

As before, $g, h \in \sqrt{m}\mathcal{P}_A$, $A = A(\omega_2) = \{n \in [1, N] \mid \xi_n(\omega_2) = 1\}$, see (4.55). One concludes that, for $\varepsilon > 0$ small and $q < \infty$ large depending on ε,

$$\begin{aligned} \log N_2(\mathcal{E}(\omega_2), t) &\le \log N_q\Big(\mathcal{P}_{A(\omega_2)}, N^{-\varepsilon} m^{-\frac12} J_{2(p-2),N}^{-\frac12}\, t\Big) \\ &\le C_q\, m \begin{cases} 1 + \log \frac1t & \text{if } 0 < t < N^\varepsilon \sqrt{m\, J_{2(p-2),N}(\omega_2)} \\ (m^{-\frac12} J_{2(p-2),N}^{-\frac12}(\omega_2)\, N^{-\varepsilon} t)^{-\nu} & \text{if } t > N^\varepsilon \sqrt{m\, J_{2(p-2),N}(\omega_2)} \end{cases} \end{aligned}$$

where $\nu > 2$, see Corollary 4.9 and Corollary 4.10. Inserting this estimate into the last term of (4.63) yields by the random majorant property on $2(p-2) \ge 2$,

$$\mathbb{E}'_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)}\, dt \le C_\varepsilon\, N^\varepsilon\, m \sqrt{\mathbb{E}\, I_{2(p-2),N}} \tag{4.65}$$

and therefore finally, by Lemma 4.3,

$$\begin{aligned} &\mathbb{E}_\omega \sup_{|a_n|\le 1} \int_0^1 \Big| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \Big|^p d\theta \\ &\le C_\varepsilon\, N^\varepsilon \Big[m^{\frac p2} + m^{\frac12}\, \mathbb{E}\, I_{p-1,N} + (1 + \sqrt{m\tau}) \sqrt{\mathbb{E}\, I_{2(p-1),N}} + m \sqrt{\mathbb{E}\, I_{2(p-2),N}}\ \Big] \\ &\le C_\varepsilon\, N^\varepsilon \Big[(\tau N)^{\frac p2} + (\tau N)^{\frac12} \Big(\tau^{p-1} N^{p-2} + (\tau N)^{\frac{p-1}{2}} \Big) \\ &\quad + (1 + \tau\sqrt{N}) \Big(\tau^{2(p-1)} N^{2p-3} + (\tau N)^{p-1} \Big)^{\frac12} + \tau N \Big(\tau^{2(p-2)} N^{2p-5} + (\tau N)^{p-2} \Big)^{\frac12} \Big] \\ &\le C_\varepsilon\, N^\varepsilon \Big[\tau^p N^{p-1} + \tau^{p-1} N^{p-\frac32} + (\tau N)^{\frac p2} \Big]. \end{aligned} \tag{4.66}$$

If $\tau \ge N^{-\frac12}$, then $\tau^p N^{p-1} \ge \tau^{p-1} N^{p-\frac32}$. Moreover, if $\tau \le N^{\frac{3-p}{p-2}}$, then $\tau^{p-1} N^{p-\frac32} \le (\tau N)^{\frac p2}$. In particular, if $3 \le p \le 4$, then $\tau^{p-1} N^{p-\frac32} \lesssim \mathbb{E}\, I_{p,N}$, and the result follows. On the other hand, if $p \ge 4$, then $\tau \ge N^{-\frac12}$ insures that $\tau^{p-1} N^{p-\frac32} \lesssim \tau^p N^{p-1} \lesssim \mathbb{E}\, I_{p,N}$, as claimed.
It remains to discuss $2 \le p \le 3$. In that case, Lemma 4.11 implies that

$$\begin{aligned} 2^{-p} \int_0^1 \Big| \sum_{n=1}^N a_n \xi_n e(n\theta) \Big|^p d\theta \ &= \int\!\!\int_0^1 \sum_{n\in R_t^1} a_n \xi_n e(n\theta) \sum_{k\in R_t^2} \bar a_k \xi_k e(-k\theta) \Big| \sum_{\ell\in R_t^2} a_\ell \xi_\ell e(\ell\theta) \Big|^{p-2} d\theta\, dt \\ &\quad + O\left(m^{\frac p2} \int_0^1 \Big(1 + \Big| \sum_{n=1}^N \frac{a_n}{\sqrt m} \xi_n e(n\theta) \Big|^2 \Big) d\theta \right). \end{aligned} \tag{4.67}$$

The integral in (4.67) is $O(1)$. Hence (4.61) changes to

$$\mathbb{E}_\omega \sup_{|a_n|\le 1}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n(\omega)e(n\theta)\Big|^p\,d\theta \lesssim m^{\frac p2} + \mathbb{E}'_{\omega_2}\,\mathbb{E}'_{\omega_1} \sup_{x\in\mathcal{E}(\omega_2)}\ \sup_{|A|=m}\sum_{n\in A}\xi_n(\omega_1)x_n, \tag{4.68}$$

with the same $\mathcal{E}(\omega_2)$, and (4.63) becomes

$$\begin{aligned}&\mathbb{E}_\omega \sup_{|a_n|\le 1}\int_0^1 \Big|\sum_{n=1}^N a_n\xi_n(\omega)e(n\theta)\Big|^p\,d\theta\\ &\le C_\varepsilon\, N^\varepsilon\Big[m^{\frac p2} + (1+\sqrt{m\tau})\sqrt{\mathbb{E}\, I_{2(p-1),N}} + \mathbb{E}'_{\omega_2}\int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\,dt\Big].\end{aligned} \tag{4.69}$$

Finally, the entropy estimate simplifies as $2(p-2)\le 2$ in this case: If $g|g|^{p-2}, h|h|^{p-2}\in\mathcal{E}(\omega_2)$, then $g,h\in\mathcal{P}_{A(\omega_2)}$ and thus

$$\begin{aligned}\|g|g|^{p-2}-h|h|^{p-2}\|_2 &\lesssim \|g-h\|_\infty\big(\|g\|_{2(p-2)}^{p-2}+\|h\|_{2(p-2)}^{p-2}\big)\\ &\le C_\varepsilon\, N^\varepsilon\|g-h\|_q\big(\|g\|_2^{p-2}+\|h\|_2^{p-2}\big)\\ &\le C_\varepsilon\, N^\varepsilon\, m^{\frac{p-2}{2}}\|g-h\|_q,\end{aligned}$$

so that now

$$\mathbb{E}'_{\omega_2}\int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\,dt \le C_\varepsilon\, N^\varepsilon\, m^{\frac p2}.$$

We leave it to the reader to check that this again leads to (4.66). As already mentioned above, the term $\tau^{p-1}N^{p-\frac32}$ can be absorbed into $(\tau N)^{\frac p2}$, since $p\le 3$. ☐

This lemma quickly leads to a proof of Proposition 4.6 in case $\delta\le\frac12$ for $p>4$, and for all $0<\delta<1$ if $2<p<4$.

Corollary 4.14. *Suppose* $0<\delta\le\frac12$ *and assume otherwise that the hypotheses of Proposition 4.6 are satisfied. Then* (4.58) *holds for all* $p\ge 4$. *If* $2<p<4$, *then* (4.58) *holds for all* $0<\delta<1$. *In particular, Proposition 4.6 is valid in these cases.*

Proof. As a first step, note that Lemma 4.13 immediately implies that all odd integers satisfy (4.58). Next, one checks that (4.58) holds at $p=\frac52$ since $2(p-1)=3$ in that case. Now Lemma 4.13 implies that (4.58) holds at all other values $p=\frac{2\ell+1}{2}$, for all integers $\ell\ge 3$. Generally speaking, one checks by means of induction that (4.58) holds at all

$$p\in\Big\{2+\frac{\ell}{2^j}\ \Big|\ \ell\in\mathbb{Z}^+\Big\} =: \mathcal{P}_j.$$

Indeed, we just verified that this holds for $j=0,1$. Now assume that it holds up to some integer j and we will prove it for $j+1$. Thus take $p=2+\frac{\ell}{2^{j+1}}\in\mathcal{P}_{j+1}$ such that $2<p<3$. Then $2(p-1)=2+\frac{\ell}{2^j}$ for which (4.58) holds by assumption. Hence Lemma 4.13 applies. Now suppose $p\in\mathcal{P}_{j+1}$ is such that $3<p<4$. Then (4.58) holds at $p-1$ by what we just did, and at $2(p-1), 2(p-2)$ by assumption. Hence Lemma 4.13 applies again. One now continues with $4<p<5$ etc., and we are done. Given any $\varepsilon > 0$ and $p > 2$ one can find $p_1 < p < p_2$ with $p_1, p_2 \in \mathcal{P}_j$ where $p_2 - p_1 < \varepsilon$. Hence (4.58) holds for all p by interpolation, as desired. It remains to deal with $\delta > \frac{1}{2}$ if $2 < p < 4$. Fix such a p. Then by Bourgain's theorem on random $\Lambda(p)$ sets, $\delta > \frac{1}{2}$ implies that the random set $S(\omega)$ is a $\Lambda(p)$ set. More precisely,

$$\mathbb{E} \sup_{|a_n|\le 1} \Big\| \sum_{n=1}^{N} a_n \xi_n(\omega) e(n\cdot) \Big\|_p^p \lesssim (\tau N)^{\frac{p}{2}}.$$

Clearly,

$$\Big\| \sum_{n=1}^{N} \xi_n(\omega) e(n\cdot) \Big\|_p \ge \Big\| \sum_{n=1}^{N} \xi_n(\omega) e(n\cdot) \Big\|_2 = \#(S(\omega))^{\frac{1}{2}},$$

and we are done. □

4.8 Choosing subsets by means of correlated selectors

To conclude this section, we want to address the issue of obtaining a version of Proposition 4.6 for subsets which are obtained by means of selectors ξ_j that are allowed to have some degree of dependence. More precisely, we will work with the selectors from the following definition.

Definition 4.15. Let $0 < \tau < 1$ be fixed. Define $\xi_j(\omega) = \chi_{[0,\tau]}(2^j \omega)$ for $j \ge 1$. Here $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with probability measure $\mathbb{P}(d\omega) = d\omega$ equal to normalized Lebesgue measure.

Since the doubling map $\omega \mapsto 2\omega \mod 1$ is measure preserving, it follows that $\mathbb{E}\,\xi_j = \tau$ and $\mathbb{P}[\xi = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$, as in the random case. However, these selector variables are no longer independent. Nevertheless, they are close enough to being independent to make the following theorem accessible to the methods of the previous section.

Theorem 4.16. *Let $0 < \delta < 1$ be fixed. For every positive integer N we let $\xi_j = \chi_{[0,\tau]}(2^j\omega)$ be as in Definition 4.15 with $\tau = N^{-\delta}$. Define a subset*

$$S(\omega) = \{ j \in [1, N] \mid \xi_j(\omega) = 1 \} \tag{4.70}$$

for every $\omega \in \mathbb{T}$. Then for every $\varepsilon > 0$ and $7 \ge p \ge 2$ one has

$$\mathbb{P}\Big[\sup_{|a_n|\le 1} \Big\| \sum_{n\in S(\omega)} a_n e(n\theta) \Big\|_{L^p(\mathbb{T})} \ge N^{\varepsilon} \Big\| \sum_{n \in S(\omega)} e(n\theta) \Big\|_{L^p(\mathbb{T})} \Big] \to 0 \tag{4.71}$$

as $N \to \infty$. Moreover, under the additional restriction $\delta \le \frac{1}{2}$, (4.71) holds for all $p \ge 7$.

To prove this theorem we may of course assume that $\tau = 2^{-k}$ for some positive integer k. Then ξ_j is measurable with respect to the dyadic intervals of length 2^{-k-j} on the unit interval $\mathbb{T}$, denoted by $\mathcal{D}_{j+k}$. Moreover, it is easy to see that ξ_j and ξ_{j+ak} are independent variables.

Lemma 4.17. *Fix $j \ge 0$ and $k \ge 1$. Let $\tau = 2^{-k}$ and ξ_i be as in Definition 4.15. Then the sequence $\left\{\xi_{j+ak}\right\}_{a=1}^{\infty}$ is a realization of a $0,1$-valued Bernoulli sequence with $\mathbb{E}\,\xi_i = \tau$.*

Proof. Fix $a > 1$ and note that the variable $\xi_{j+ak}(\omega)$ is $2^{-(j+ak)}$-periodic. On the other hand, each of the variables ξ_{j+bk} with $b < a$ is constant on intervals from $\mathcal{D}_{j+ak}$ (which is the same as saying that these variables are all $\mathcal{D}_{j+ak}$ measurable). It follows that

$$\mathbb{P}\big[\xi_{j+ak} = 1 \mid \xi_{j+bk} = \varepsilon_b,\ 0 \le b \le a-1\big] = \tau = \mathbb{P}[\xi_{j+ak} = 1],$$

for any choice of $\varepsilon_b = 0, 1,\ 0 \le b \le a-1$. This implies independence. □

From now on, let $\tau = N^{-\delta}$ for some fixed $0 < \delta < 1$. In view of Lemma 4.17 we can decompose the sequence $\{\xi_j\}_{j=1}^N$ into about $\log N$ many subsequences, where the indices run along arithmetic progressions $\mathcal{P}_i$ of step-size equal to $\sim \log N$, and $1 \le i \lesssim \log N$. Each of the subsequences consists of i.i.d. variables, but variables from different subsequences are not independent. This easily shows that Lemma 4.3 remains valid here, possibly with a logarithmic loss in the upper bound for $\mathbb{E}\, I_{p,N}$. Indeed, recall that the proof of that Lemma is based upon splitting a random trigonometric polynomial into its expectation and a mean-zero part. Since the L^p-norm of the Dirichlet kernel on an arithmetic progression of length K is about $K^{\frac{1}{p'}}$, and here $\#\mathcal{P}_i \sim \frac{N}{\log N}$, one sees immediately that the upper bound from (4.7) is the same up to logarithmic factors. As far as the lower bound of Lemma 4.3 is concerned, note that the proof relies on obtaining *upper* bounds on certain error terms, cf. (4.10)-(4.13). However, these upper bounds are again immediate corollaries of the random case by virtue of the splitting into the progressions $\mathcal{P}_i$.

The consequence of this is that basically all the main estimates from the previous section remain valid here, up to possibly an extra factor of $\log N$. Clearly, such factors are irrelevant in this context. More precisely, with ξ_j as in Definition 4.15 and $S(\omega)$ as in (4.70), it is a corollary of the proof of Proposition 4.6 that

$$\mathbb{E} \sup_{|a_n| \le 1} \Big\| \sum_{n \in S(\omega)} a_n e(n\theta) \Big\|_{L^p(\mathbb{T})}^p \le C_\varepsilon\, N^\varepsilon \big(\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}}\big). \tag{4.72}$$

The proof of Theorem 4.16 is therefore completed as before by appealing to (the adapted version) of Lemma 4.3.

Remark 4.18. Other examples of much more strongly correlated selectors are $\xi_j(\omega) = \chi_{[0,\tau]}(j^s\omega)$ where s is a fixed positive integer and $\omega \in \mathbb{T}$. It appears to be rather difficult to prove a version of Proposition 4.6 for these types of selectors.

5 Perturbing arithmetic progressions

Let $\mathcal{P} \subset [1, N]$ be an arithmetic progression of length L, i.e.,

$$\mathcal{P} = \{b + a\ell \mid 0 < b < a,\ 0 \le \ell < L := \lfloor N/a \rfloor\} \subset [1, N].$$

Fix some arbitrary $\varepsilon_0 > 0$. Suppose $N^{\varepsilon_0} < s < a$ and let $\{\xi_j\}_{j \in \mathcal{P}}$ be i.i.d. variables, integer valued and uniformly distributed in $[-s, s]$. We define a random subset

$$\mathcal{S}(\omega) := \{j + \xi_j(\omega) \mid j \in \mathcal{P}\}. \tag{5.1}$$

For future reference, we set $I_j := [j - s, j + s]$ for each $j \in \mathcal{P}$. By construction, $S(\omega) \subset \bigcup_{j \in \mathcal{P}} I_j$, and the intervals I_j are congruent and pairwise disjoint.

5.1 Suprema of random processes

The following lemma is related to Lemma 4.7.

Lemma 5.1. *Let $\mathcal{E} \subset \mathbb{R}_+^N$, $B = \sup_{x\in\mathcal{E}} |x|$, and $\mathcal{S}(\omega)$ be as in* (5.1)*. Then*

$$\mathbb{E}_\omega \sup_{x\in\mathcal{E}} \sum_{j\in\mathcal{S}(\omega)} x_j \lesssim B\big(1+\sqrt{L/s}\big) + \int_0^B \sqrt{\log N_2(\mathcal{E},t)}\,dt$$

where N_2 refers to the L^2 entropy.

Proof. As in the proof of Lemma 4.7, we introduce 2^{-k}-nets $\mathcal{E}_k$ and $\mathcal{F}_k \subset \mathbb{R}^N$ so that $\operatorname{diam}(\mathcal{F}_k) \le 1$,

$$\log \#\mathcal{F}_k \le C \log \#\mathcal{E}_{k+1}, \tag{5.2}$$

and

$$\mathbb{E} \sup_{x\in\mathcal{E}} \sum_{n\in\mathcal{S}(\omega)} x_n \le \sum_{k\ge k_0} 2^{-k+1}\, \mathbb{E} \sup_{y\in\mathcal{F}_k} \sum_{n\in\mathcal{S}(\omega)} |y_n|. \tag{5.3}$$

Now fix some $k \ge k_0$ and write $\mathcal{F}$ instead of $\mathcal{F}_k$. With $0 < \rho_2$ to be determined, one has for any $|y| \le 1$

$$\sum_{i\in\mathcal{S}(\omega)} y_i \le \sum_{y_i\ge\rho_2} y_i + \sum_{y_i<\rho_2} \chi_{\mathcal{S}(\omega)}(i)\, y_i \le \rho_2^{-1} + \sum_{y_i<\rho_2} \chi_{\mathcal{S}(\omega)}(i)\, y_i.$$

Let $q := 1 + \lfloor \log \mathcal{F} \rfloor$. Then, as in (4.24),

$$\mathbb{E} \sup_{y\in\mathcal{F}} \sum_{i\in\mathcal{S}(\omega)} y_i \lesssim \rho_2^{-1} + \sup_{|y|\le 1} \Big\| \sum_{y_i<\rho_2} \chi_{\mathcal{S}(\omega)}(i)\, y_i \Big\|_{L^q(\omega)}. \tag{5.4}$$

To control the last term in (5.4), we need the following analogue of (4.25). By the multinomial theorem (for any positive integer q),

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{n\in\mathcal{S}(\omega)} \chi_A(n)\Big]^q = \mathbb{E}\Big[\sum_{j\in\mathcal{P}} \chi_A(j+\xi_j(\omega))\Big]^q = \sum_{q_1+\ldots+q_L=q} \binom{q}{q_1,\ldots,q_L} \mathbb{E} \prod_{j\in\mathcal{P}} \chi_A(j+\xi_j(\omega))^{q_j} \\
&= \sum_{\nu=1}^{q} \sum_{\substack{1\le i_1<i_2<\ldots<i_\nu\le L\\ i_1,\ldots,i_\nu\in\mathcal{P}}} \sum_{\substack{q_{i_1}+\ldots+q_{i_\nu}=q\\ q_{i_1}\ge1,\ldots,q_{i_\nu}\ge1}} \binom{q}{q_{i_1},\ldots,q_{i_\nu}} \mathbb{E} \prod_{t=1}^{\nu} \chi_A(i_t+\xi_{i_t}(\omega)) \\
&= \sum_{\nu=1}^{q} \sum_{\substack{1\le i_1<i_2<\ldots<i_\nu\le L\\ i_1,\ldots,i_\nu\in\mathcal{P}}} \sum_{\substack{q_{i_1}+\ldots+q_{i_\nu}=q\\ q_{i_1}\ge1,\ldots,q_{i_\nu}\ge1}} \binom{q}{q_{i_1},\ldots,q_{i_\nu}} \frac{|A\cap I_{i_1}|}{|I_{i_1}|}\frac{|A\cap I_{i_2}|}{|I_{i_2}|}\cdot\ldots\cdot\frac{|A\cap I_{i_\nu}|}{|I_{i_\nu}|} \\
&\le \sum_{\nu=1}^{q} \nu^q \frac{1}{\nu!}\Big(\sum_{j\in\mathcal{P}} \frac{|A\cap I_j|}{|I_j|}\Big)^\nu \le \sum_{\nu=1}^{q} \frac{\nu^q}{q!}\frac{q!}{\nu!}\Big(\frac{|A\cap \bigcup_{j\in\mathcal{P}} I_j|}{2s+1}\Big)^\nu \\
&\le \sum_{\nu=1}^{q} \binom{q}{\nu} q^{q-\nu} \Big(\frac{e|A\cap \bigcup_{j\in\mathcal{P}} I_j|}{2s+1}\Big)^\nu \le \Big(q + \frac{e|A\cap \bigcup_{j\in\mathcal{P}} I_j|}{2s+1}\Big)^q.
\end{aligned}$$

Continuing with the final term in (5.4) one concludes that

$$\begin{aligned}\sup_{|y|\le 1}\Big\|\sum_{y_i<\rho_2}\chi_{\mathcal{S}(\omega)}(i)\,y_i\Big\|_{L^q(\omega)} &\lesssim \sum_{\rho_2^{-2}<2^j} 2^{-\frac{j}{2}}\sup_{|A|=2^j}\Big\|\sum_{n\in\mathcal{S}(\omega)}\chi_A(n)\Big\|_{L^q(\omega)}\\ &\lesssim \sum_{\rho_2^{-2}<2^j} 2^{-\frac{j}{2}}\Big(q+\frac{\min(Ls,2^j)}{s}\Big)\lesssim q\rho_2+\sqrt{L/s}.\end{aligned}$$

Let $\rho_2 = q^{-\frac{1}{2}} = (1+\log\#\mathcal{F})^{-\frac{1}{2}}$. Inserting this bound into (5.4) therefore yields

$$\mathbb{E}\sup_{y\in\mathcal{F}}\sum_{i\in\mathcal{S}(\omega)} y_i \lesssim \sqrt{q}+\sqrt{L/s}\lesssim\sqrt{L/s}+1+\sqrt{\log\#\mathcal{F}}.$$

The lemma now follows in view of (5.2) and (5.3). □

5.2 The L^p norm of the Dirichlet kernel over $\mathcal{S}(\omega)$

The following lemma determines an upper bound on the typical size of the Dirichlet kernel over $\mathcal{S}(\omega)$ in the L^p-norm, with $2\le p\le 4$. The lower bound, as well as the case $p>4$ will be dealt with below.

Lemma 5.2. *With $\mathcal{S}(\omega)$ as in* (5.1), *there exists a constant C_p so that*

$$\mathbb{E}\Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\cdot)\Big\|_p^p \le C_p\Big(L^{\frac{p}{2}}+\frac{L^{p-1}}{s}\Big)$$

for all $2\le p\le 4$.

Proof. For every $\ell\in\mathbb{Z}$ define

$$A_\ell(\omega) := \#\{n,m\in\mathcal{S}(\omega)\mid n-m=\ell\} = \sum_{j,k\in\mathcal{P}}\chi_{[j-k+\xi_j-\xi_k=\ell]}.$$

Clearly, $\mathcal{P}-\mathcal{P}\subset\bigcup_i J_i$ where $i\in a\mathbb{Z}$ and $J_i := [i-2s, i+2s]$. These intervals are mutually disjoint since $s\ll a$. This means that

$$\ell\in J_i \implies A_\ell(\omega)=\sum_{j\in\mathcal{P}}\chi_{[j-i\in\mathcal{P}]}\chi_{[\xi_j-\xi_{j-i}=\ell-i]}.$$

Let us denote the unique i for which $\ell\in J_i$ by $i(\ell)$. For simplicity, we shall mostly write i. If $i=0$, then $A_\ell(\omega)=L\delta_0(\ell)$ (recall that $\#\mathcal{P}=L$). Otherwise, if $i\neq 0$, then one finds that

$$\begin{aligned}(5.5)\qquad \mathbb{E}\,A_\ell &= \sum_{j\in\mathcal{P}}\frac{2}{2s+1}\Big(1-\frac{|\ell-i|}{s}\Big)_+\chi_{\mathcal{P}}(j-i) = (L-|i|/a)_+\frac{2}{2s+1}\Big(1-\frac{|\ell-i|}{s}\Big)_+\\ (5.6)\qquad &= \frac{2L}{2s+1}\widehat{K_s}(\ell-i(\ell))\widehat{K_L}(|i|/a)\end{aligned}$$

where $\widehat{K_n}(k) = (1-|k|/n)_+$ denotes the Fejer kernel. Moreover, if $i \neq 0$, then

$$\begin{aligned}
\mathbb{E}\, A_\ell^2 &= \mathbb{E} \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \chi_{[\xi_j-\xi_{j-i}=\ell-i]}\chi_{[\xi_k-\xi_{k-i}=\ell-i]} \\
&= \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \chi_{[j\neq k, j\neq k\pm i]}\, \mathbb{E}\,\chi_{[\xi_j-\xi_{j-i}=\ell-i]}\, \mathbb{E}\,\chi_{[\xi_k-\xi_{k-i}=\ell-i]} \\
&\quad + \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \Big(\chi_{[j=k, j\neq k\pm i]} + \chi_{[j\neq k, k+i, j=k-i]} + \chi_{[j\neq k, k-i, j=k+i]}\Big)\, \mathbb{E}\,\chi_{[\xi_j-\xi_{j-i}=\ell-i]}\chi_{[\xi_k-\xi_{k-i}=\ell-i]}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\, A_\ell^2 &= \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \mathbb{E}\,\chi_{[\xi_j-\xi_{j-i}=\ell-i]}\, \mathbb{E}\,\chi_{[\xi_k-\xi_{k-i}=\ell-i]} \\
(5.7)\quad &+ \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \Big(\chi_{[j=k, j\neq k\pm i]} + \chi_{[j\neq k, k+i, j=k-i]} + \chi_{[j\neq k, k-i, j=k+i]}\Big)\, \mathbb{E}\,\Big(\chi_{[\xi_j-\xi_{j-i}=\ell-i]}\chi_{[\xi_k-\xi_{k-i}=\ell-i]}\Big) \\
(5.8)\quad &- \sum_{\substack{j,k\in\mathcal{P} \\ j-i\in\mathcal{P},\, k-i\in\mathcal{P}}} \Big(\chi_{[j=k, j\neq k\pm i]} + \chi_{[j\neq k, k+i, j=k-i]} + \chi_{[j\neq k, k-i, j=k+i]}\Big)\, \mathbb{E}\,\chi_{[\xi_j-\xi_{j-i}=\ell-i]}\, \mathbb{E}\,\chi_{[\xi_k-\xi_{k-i}=\ell-i]} \\
(5.9)\quad &= (\mathbb{E}\, A_\ell)^2 + O\Big(\frac{L}{s}\Big(1-\frac{|\ell-i|}{s}\Big)_+\Big).
\end{aligned}$$

The O-term in (5.9) arises because the error terms in (5.7) and (5.8) basically reduce to the computation of a single expectation as in (5.5). Now consider

$$V_{p,N} := \int_0^1 \Big| \sum_{\ell\in\mathbb{Z}} (A_\ell(\omega) - \mathbb{E}\, A_\ell) e(\ell\theta) \Big|^{\frac{p}{2}}\, d\theta.$$

Since $p \leq 4$ by assumption, $\mathbb{E}\, V_{p,N} \leq (\mathbb{E}\, V_{4,N})^{\frac{p}{4}}$. Moreover, by (5.9),

$$\begin{aligned}
\mathbb{E}\, V_{4,N} &= \mathbb{E} \sum_{\ell\in\mathbb{Z}} |A_\ell(\omega) - \mathbb{E}\, A_\ell|^2 = \sum_{\ell\in\mathbb{Z}} \Big[\mathbb{E}\,(A_\ell^2) - (\mathbb{E}\, A_\ell)^2\Big] \\
&= \mathbb{E}\,(A_0^2) - (\mathbb{E}\, A_0)^2 + \sum_{\ell\neq 0} \Big[(\mathbb{E}\, A_\ell)^2 + O\Big(\frac{L}{s}\Big(1-\frac{|\ell-i|}{2s+1}\Big)_+\Big)\Big] - \sum_{\ell\neq 0} (\mathbb{E}\, A_\ell)^2 \\
&\lesssim L^2
\end{aligned}$$

and therefore

$$\mathbb{E}\, V_{p,N} \lesssim L^{\frac{p}{2}}. \tag{5.10}$$

In view of (5.6),

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} \mathbb{E}\, A_\ell\, e(\ell\theta) &= \sum_{\ell\in\mathbb{Z}} \frac{2L}{2s+1}\widehat{K_s}(\ell-i(\ell))\widehat{K_L}(|i(\ell)|/a)e((\ell-i(\ell))\theta)e(i(\ell)\theta)\\ &= \frac{2L}{2s+1}\sum_{k\in\mathbb{Z}}\widehat{K_s}(k)e(k\theta)\sum_{j\in\mathbb{Z}}\widehat{K_L}(j)e(ja\theta) = \frac{2L}{2s+1}K_s(\theta)K_L(a\theta).\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^1 \Big|\sum_{\ell\in\mathbb{Z}} \mathbb{E}\, A_\ell\, e(\ell\theta)\Big|^{\frac{p}{2}} d\theta &\lesssim \Big(\frac{L}{s}\Big)^{\frac{p}{2}} \int_0^1 \Big(\frac{1}{s}\min(s^2,\theta^{-2})\Big)^{\frac{p}{2}} |K_L(a\theta)|^{\frac{p}{2}}\, d\theta\\ &\lesssim \Big(\frac{L}{s}\Big)^{\frac{p}{2}}\Big\{s^{\frac{p}{2}}\, a^{-1}\, L^{\frac{p}{2}-1} + \sum_{j=1}^{a}\Big(\frac{1}{s}\min(s^2,(j/a)^{-2})\Big)^{\frac{p}{2}} a^{-1}\, L^{\frac{p}{2}-1}\Big\}\\ &\lesssim \frac{L^{p-1}}{s}. \qquad (5.11)\end{aligned}$$

Combining (5.10) with (5.11) one obtains for $2\le p\le 4$

$$\begin{aligned}&\mathbb{E}\int_0^1\Big|\sum_{n\in\mathcal{S}(\omega)} e(n\theta)\Big|^p d\theta = \mathbb{E}\int_0^1\Big|\sum_{\ell\in\mathbb{Z}} A_\ell(\omega)e(\ell\theta)\Big|^{\frac{p}{2}} d\theta\\ &\lesssim \int_0^1\Big|\sum_{\ell\in\mathbb{Z}} \mathbb{E}\, A_\ell\, e(\ell\theta)\Big|^{\frac{p}{2}} d\theta + \mathbb{E}\int_0^1\Big|\sum_{\ell\in\mathbb{Z}}[A_\ell(\omega) - \mathbb{E}\, A_\ell]e(\ell\theta)\Big|^{\frac{p}{2}} d\theta\\ &\lesssim \frac{L^{p-1}}{s} + L^{\frac{p}{2}}, \qquad (5.12)\end{aligned}$$

as claimed. □

The following lemma is a special case of a well-known large deviation estimate for martingales with bounded increments. The norm $\|\cdot\|_\infty$ refers to the supremum norm with respect to the probability space.

Lemma 5.3. *Suppose $\{X_j\}_{j=1}^M$ are complex-valued independent variables with $\mathbb{E}\, X_j = 0$. Then for all $\lambda > 0$*

$$\mathbb{P}\Big[\big|\sum_{j=1}^M X_j\big| > \lambda\Big(\sum_{j=1}^M \|X_j\|_\infty^2\Big)^{\frac{1}{2}}\Big] < C\, e^{-c\lambda^2}$$

with some absolute constants c, C.

Lemma 5.3 implies the following simple generalization of the Salem-Zygmund bound.

Corollary 5.4. *Let s, L be positive integers. Suppose T_L is a trigonometric polynomial with random coefficients that can be written in the form*

$$T_L(\theta) = \sum_{j=-L}^{L} a_j(\theta)e(j\theta)$$

where $a_j(\theta)$ are trigonometric polynomials of degree at most s, and such that for fixed θ they are independent random variables with $\mathbb{E}\, a_j(\theta) = 0$. Moreover, we assume that $\sup_{\theta\in\mathbb{T}} |a_j(\theta)| \le 1$ for each j. Then for every $A > 1$

$$\mathbb{P}[\|T_L\|_\infty > C\sqrt{\log(s+L)}\,\sqrt{L}] \le (s+L)^{-A},$$

with some constant $C = C(A)$.

Proof. Fix $\theta \in \mathbb{T}$ and apply Lemma 5.3 with $X_j = a_j(\theta)e(j\theta)$. By assumption, these are complex valued independent mean-zero variables with $\|X_j\|_\infty \le 1$. Therefore,

$$\sup_{\theta\in\mathbb{T}} \mathbb{P}\Big[\big|\sum_{j=-L}^{L} a_j(\theta)e(j\theta)\big| > \lambda\sqrt{L}\Big] < C\, e^{-c\lambda^2}. \tag{5.13}$$

If $|\theta - \theta'| < (s+L)^{-2}$, then by Bernstein's inequality

$$|T_L(\theta) - T_L(\theta')| \le (s+L)\|T_L\|_\infty|\theta-\theta'| \lesssim (s+L)L(s+L)^{-2} \lesssim 1.$$

Now pick a $(s+L)^{-2}$-net on the circle. The corollary follows by setting $\lambda = C\log(s+L)$ with C large, and summing (5.13) over the elements of the net. □

We can now state the general version of Lemma 5.2. It is possible to remove the log-term from the upper bound, but the bound given below suffices for our purposes.

Lemma 5.5. *For all $p \ge 2$ there exists C_p so that*

$$\mathbb{E}\Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\cdot)\Big\|_p^p \le C_p\Big(\frac{L^{p-1}}{s} + (L\,\log N)^{\frac{p}{2}}\Big). \tag{5.14}$$

Moreover, there is $c_p > 0$ small so that

$$\mathbb{P}\Big[\Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\cdot)\Big\|_p^p < c_p\Big(L^{\frac{p}{2}} + \frac{L^{p-1}}{s}\Big)\Big] \to 0$$

as $N \to \infty$.

Proof. We work with the following splitting:

$$\sum_{n\in\mathcal{S}(\omega)} e(n\theta) = \sum_{n\in\mathbb{Z}} \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta) + \sum_{n\in\mathbb{Z}} \Big[\chi_{\mathcal{S}(\omega)}(n) - \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta). \tag{5.15}$$

Clearly,

$$\sum_{n\in\mathbb{Z}} \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta) = \frac{1}{2s+1}D_s(\theta)\sum_{j\in\mathcal{P}} e(j\theta), \tag{5.16}$$

and thus

$$\Big\|\sum_{n\in\mathbb{Z}}\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta)\Big\|_p^p \;\lesssim\; s^{-p}\int_0^1\Big|\min(s,\theta^{-1})\sum_{j=1}^{L}e(ja\theta)\Big|^p\,d\theta$$

$$\lesssim\; s^{-p}\Big[\sum_{k=1}^{L}\min(s,a/k)^p+s^p\Big]\frac{L^{p-1}}{a}\lesssim\frac{L^{p-1}}{s}. \tag{5.17}$$

Conversely,

$$\Big\|\sum_{n\in\mathbb{Z}}\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta)\Big\|_p^p \;\gtrsim\; s^{-p}\int_0^{1/s}\Big|D_s(\theta)\sum_{j=1}^{L}e(ja\theta)\Big|^p\,d\theta$$

$$\gtrsim\; \frac{a}{s}\frac{L^{p-1}}{a}=\frac{L^{p-1}}{s}. \tag{5.18}$$

Both (5.17) and (5.18) hold for all $p>1$. The second sum in (5.15) can be written as

$$\sum_{n\in\mathbb{Z}}\Big[\chi_{\mathcal{S}(\omega)}(n)-\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta)=\sum_{j\in\mathcal{P}}a_j(\omega,\theta)e(j\theta),$$

where $a_j(\omega,\theta)=\chi_{I_j}(\xi_j(\omega))e(\xi_j(\omega))-\frac{1}{2s+1}D_s(\theta)$. Clearly, $\mathbb{E}\,a_j(\omega,\theta)=0$, $\sup_\theta|a_j(\omega,\theta)|\le 2$ and for fixed θ the random variables $a_j(\omega,\theta)$ are independent. Thus Corollary 5.4 yields that

$$\Big\|\sum_{n\in\mathbb{Z}}\Big[\chi_{\mathcal{S}(\omega)}(n)-\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta)\Big\|_\infty\lesssim\sqrt{L}\sqrt{\log N} \tag{5.19}$$

up to probability at most $(s+L)^{-p}$. In particular,

$$\mathbb{E}\Big\|\sum_{n\in\mathbb{Z}}\Big[\chi_{\mathcal{S}(\omega)}(n)-\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta)\Big\|_p^p\lesssim(L\,\log N)^{\frac{p}{2}}+L^p(s+L)^{-p}\lesssim(L\,\log N)^{\frac{p}{2}}.$$

In conjunction with (5.17) this yields (5.14). For the lower bound, take $N^{-\varepsilon_0/2}>h\gg\frac{1}{s}$. Then

$$\int_0^1\Big|\sum_{n\in\mathcal{S}(\omega)}e(n\theta)\Big|^p\,d\theta\gtrsim\int_0^{1/s}\Big|\sum_{n\in\mathbb{Z}}\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta)\Big|^p\,d\theta-\int_0^{1/s}\Big|\sum_{n\in\mathbb{Z}}\Big[\chi_{\mathcal{S}(\omega)}(n)-\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta)\Big|^p\,d\theta$$

$$+\int_h^{1-h}\Big|\sum_{n\in\mathbb{Z}}\Big[\chi_{\mathcal{S}(\omega)}(n)-\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big]e(n\theta)\Big|^p\,d\theta-\int_h^{1-h}\Big|\sum_{n\in\mathbb{Z}}\mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)e(n\theta)\Big|^p\,d\theta$$

$$=: I+II+III+IV. \tag{5.20}$$

By (5.18), $I\gtrsim\frac{L^{p-1}}{s}$. Secondly,

$$IV\;\lesssim\;\int_h^{1-h}\Big|\frac{1}{s}D_s(\theta)\sum_{j=0}^{L-1}e(ja\theta)\Big|^p\,d\theta$$

$$\lesssim\; s^{-p}\sum_{j>ah}(j/a)^{-p}\,\frac{L^{p-1}}{a}\lesssim s^{-p}h^{-p+1}L^{p-1}\ll\frac{L^{p-1}}{s} \tag{5.21}$$

where the final estimate follows from $hs \gg 1$. Thirdly, in view of $p \geq 2$ and (5.19),

$$\begin{aligned} III &\gtrsim \int_0^1 \Big|\sum_{n\in\mathbb{Z}} \Big[\chi_{\mathcal{S}(\omega)}(n) - \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big] e(n\theta)\Big|^p\, d\theta \\ &\qquad - \Big(\int_0^h + \int_{1-h}^1\Big)\Big|\sum_{n\in\mathbb{Z}} \Big[\chi_{\mathcal{S}(\omega)}(n) - \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big] e(n\theta)\Big|^p\, d\theta \\ &\gtrsim \Big(\int_0^1 \Big|\sum_{n\in\mathbb{Z}} \Big[\chi_{\mathcal{S}(\omega)}(n) - \mathbb{E}\,\chi_{\mathcal{S}(\omega)}(n)\Big] e(n\theta)\Big|^2\, d\theta\Big)^{\frac{p}{2}} - C\,h(L\,\log N)^{\frac{p}{2}} \\ &\gtrsim L^{\frac{p}{2}} - C\,h(L\,\log N)^{\frac{p}{2}}, \end{aligned} \tag{5.22}$$

up to probability $(s+L)^{-p} = o(1)$ as $N \to \infty$. Similarly, (5.19) implies that

$$II \lesssim s^{-1}\,(L\,\log N)^{\frac{p}{2}}$$

up to probability $(s+L)^{-p}$. Combining this bound with (5.22), (5.21), and (5.20) implies that

$$\int_0^1 \Big|\sum_{n\in\mathcal{S}(\omega)} e(n\theta)\Big|^p\, d\theta \gtrsim \frac{L^{p-1}}{s} + L^{\frac{p}{2}} - C(h+s^{-1})(L\,\log N)^{\frac{p}{2}}$$

asymptotically with probability one. Since $h < N^{-\varepsilon}$ and $s > N^{\varepsilon}$, the lemma follows. □

5.3 The majorant property for randomly perturbed arithmetic progressions

We are now ready to state our first result for perturbed arithmetic progressions as defined in (5.1). In this section, if $\mathcal{S}$ is the perturbation of an arithmetic progression of length L, then we write

$$A_{p,L}(\omega) := \Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\cdot)\Big\|_p^p.$$

Also, we say that the random majorant property (RMP) holds at p if

$$\mathbb{E}_\omega \sup_{|a_n|\leq 1} \Big\|\sum_{n\in\mathcal{S}(\omega)} a_n\, e(n\cdot)\Big\|_p^p \leq C_\varepsilon\, N^\varepsilon\, \mathbb{E}_\omega \Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\theta)\Big\|_p^p. \tag{5.23}$$

Of course, this depends on the length L of the underlying arithmetic progression. Although L is arbitrary, it will be kept fixed in the course of any argument that uses (5.23).

Theorem 5.6. *Let $\mathcal{S}$ be as in* (5.1)*. Then for every $\varepsilon > 0$ and $4 \geq p \geq 2$ one has*

$$\mathbb{P}\Big[\sup_{|a_n|\leq 1} \Big\|\sum_{n\in\mathcal{S}(\omega)} a_n e(n\theta)\Big\|_{L^p(\mathbb{T})} \geq N^\varepsilon \Big\|\sum_{n\in\mathcal{S}(\omega)} e(n\theta)\Big\|_{L^p(\mathbb{T})}\Big] \to 0 \tag{5.24}$$

as $N \to \infty$. Moreover, under the additional restriction $L \geq s$, (5.24) *holds for all $p \geq 4$.*

Proof. The proof is similar to the random case of the previous section, so we shall be somewhat brief. We will show that the RMP holds at p provided either $2 \le p \le 3$, or if the RMP holds at $p-1$, $2(p-1)$, and $2(p-2)$. It is important to notice that the RMP at p implies (5.24). Firstly, recall that we can write $\mathcal{S}(\omega) = \{j + \xi_j \mid j \in \mathcal{P}\}$. We apply the decoupling lemma, Lemma 4.11, to the progression $\mathcal{P}$. I.e., in the notation of Lemma 4.11, $R_t^1 := \{j \in \mathcal{P} \,|\, \zeta_j = 1\}$, and $R_t^2 := \{j \in \mathcal{P} \,|\, \zeta_j = 0\}$. Set

$$\mathcal{S}_t^1(\omega) := \{j + \xi_j(\omega) \mid j \in R_t^1\}, \quad \mathcal{S}_t^2(\omega) := \{j + \xi_j(\omega) \mid j \in R_t^2\}.$$

Therefore, by Lemma 4.11,

$$\begin{aligned} \frac{1}{8}\int_0^1 \Big| \sum_{n\in\mathcal{S}(\omega)} a_n e(n\theta)\Big|^p d\theta \;&=\; \int\int_0^1 \sum_{n\in\mathcal{S}_t^1(\omega)} a_n\, e(n\theta) \sum_{k\in\mathcal{S}_t^2(\omega)} \bar{a}_k\, e(-k\theta)\Big| \sum_{\ell\in\mathcal{S}_t^2(\omega)} a_\ell\, e(\ell\theta)\Big|^{p-2} d\theta\, dt \\ &\quad + O\left(L^{\frac{p}{2}} \int_0^1 \Big(1 + \Big|\sum_{n\in\mathcal{S}} \frac{a_n}{\sqrt{L}}\, e(n\theta)\Big|^{\max(p-1,2)}\Big)\, d\theta \right). \end{aligned} \tag{5.25}$$

If either $p \le 3$, or if the RMP holds at $p-1$, then the O-term in (5.25) is at most

$$L^{\frac{p}{2}} + C_\varepsilon\, N^\varepsilon L^{\frac{1}{2}}\, \mathbb{E}\, A_{p-1,L} \lesssim N^\varepsilon L^{\frac{p}{2}}, \tag{5.26}$$

see Lemma 5.5. We therefore obtain as in (4.51),

$$\begin{aligned} &\mathbb{E}_\omega \sup_{|a_n|\le 1} \int_0^1 \Big| \sum_{n\in\mathcal{S}(\omega)} a_n\, e(n\theta)\Big|^p d\theta \\ &\lesssim\; C_\varepsilon\, N^\varepsilon L^{\frac{p}{2}} + \int \mathbb{E}_{\omega_1,\omega_2} \sup_{|a_n|\le 1} \left| \int_0^1 \sum_{n\in\mathcal{S}_t^1(\omega_1)} a_n\, e(n\theta) \sum_{k\in\mathcal{S}_t^2(\omega_2)} \bar{a}_k\, e(-k\theta)\, \Big| \sum_{\ell\in\mathcal{S}_t^2(\omega_2)} a_\ell\, e(\ell\theta)\Big|^{p-2} d\theta \right| dt \\ &\lesssim\; C_\varepsilon\, N^\varepsilon L^{\frac{p}{2}} + \int \mathbb{E}_{\omega_1}\, \mathbb{E}_{\omega_2} \sup_{\substack{|a_n|\le 1\\ |b_n|\le 1}} \left| \int_0^1 \sum_{n\in\mathcal{S}_t^1(\omega_1)} a_n\, e(n\theta) \sum_{k\in\mathcal{S}_t^2(\omega_2)} \bar{b}_k\, e(-k\theta)\, \Big| \sum_{\ell\in\mathcal{S}_t^2(\omega_2)} b_\ell\, e(\ell\theta)\Big|^{p-2} d\theta \right| dt \\ &\lesssim\; C_\varepsilon\, N^\varepsilon L^{\frac{p}{2}} + \int \mathbb{E}_{\omega_1}\, \mathbb{E}_{\omega_2} \sup_{\substack{|a_n|\le 1\\ |b_n|\le 1}} \left| \int_0^1 \sum_{n\in\mathcal{S}(\omega_1)} a_n\, e(n\theta) \sum_{k\in\mathcal{S}(\omega_2)} \bar{b}_k\, e(-k\theta)\, \Big| \sum_{\ell\in\mathcal{S}(\omega_2)} b_\ell\, e(\ell\theta)\Big|^{p-2} d\theta \right| dt \\ &\lesssim C_\varepsilon\, N^\varepsilon L^{\frac{p}{2}} + \mathbb{E}_{\omega_2}\, \mathbb{E}_{\omega_1} \sup_{x\in\mathcal{E}(\omega_2)} \sum_{n\in\mathcal{S}(\omega_1)} x_n. \end{aligned} \tag{5.27}$$

Here

$$\mathcal{E}(\omega_2) := \left\{ \left(\Big| \Big\langle e(n\cdot), \sum_{k\in\mathcal{S}(\omega_2)} \bar{b}_k\, e(-k\cdot) \Big| \sum_{\ell\in\mathcal{S}(\omega_2)} b_\ell\, e(\ell\cdot)\Big|^{p-2} \Big\rangle \Big| \right)_{n=1}^N \,\Big|\, \sup_{1\le n\le N} |b_n| \le 1 \right\} \subset \mathbb{R}_+^N.$$

By Lemma 5.1, it follows from (5.27) that

$$\begin{aligned}
(5.28)\qquad & \mathbb{E}_{\omega} \sup_{|a_n|\le 1} \int_0^1 \Big| \sum_{n\in\mathcal{S}(\omega)} a_n\, e(n\theta)\Big|^p \, d\theta \\
& \lesssim C_\varepsilon\, N^\varepsilon L^{\frac{p}{2}} + (1+\sqrt{L/s})\, \mathbb{E}_{\omega_2} \sup_{x\in\mathcal{E}(\omega_2)} |x| + \mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt.
\end{aligned}$$

Now suppose the RMP holds at $2(p-1)$ (so this holds for sure if p is an odd integer). Then by Plancherel,

$$\mathbb{E}_{\omega_2} \sup_{x\in\mathcal{E}(\omega_2)} |x| \le C_\varepsilon N^\varepsilon\, \mathbb{E}_{\omega} \Big\| \sum_{n\in\mathcal{S}(\omega)} e(n\cdot)\Big\|_{2(p-1)}^{p-1} \le C_\varepsilon N^\varepsilon \sqrt{\mathbb{E}_{\omega} A_{2(p-1),L}(\omega)}.$$

As far as the entropy term in (5.28) is concerned, the same analysis as in the random case shows that if $p\le 3$, then

$$\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt \le C_\varepsilon\, N^\varepsilon\, L^{\frac{3}{2}},$$

or if $p>3$ and the RMP holds at $2(p-2)$, then

$$\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)}\, dt \le C_\varepsilon\, N^\varepsilon\, L \sqrt{\mathbb{E}\, A_{2(p-2),L}},$$

see (4.64) and (4.65) for the details. Inserting all of this into (5.28) yields, under the assumption that $p>3$ and the RMP holds at $p-1$, $2(p-1)$, and $2(p-2)$ (the case $p\le 3$ is similar),

$$\begin{aligned}
& \mathbb{E}_{\omega} \sup_{|a_n|\le 1} \int_0^1 \Big| \sum_{n\in\mathcal{S}(\omega)} a_n\, e(n\theta)\Big|^p \, d\theta \\
\lesssim\ & C_\varepsilon\, N^\varepsilon \Big\{ L^{\frac{p}{2}} + (1+\sqrt{L/s}) \sqrt{\mathbb{E}_{\omega} A_{2(p-1),L}(\omega)} + L\sqrt{\mathbb{E}\, A_{2(p-2),L}} \Big\} \\
\lesssim\ & C_\varepsilon\, N^\varepsilon \Big\{ L^{\frac{p}{2}} + (1+\sqrt{L/s}) \Big(\frac{L^{2p-3}}{s} + L^{p-1}\Big)^{\frac{1}{2}} + L\Big(\frac{L^{2p-5}}{s} + L^{p-2}\Big)^{\frac{1}{2}} \Big\} \\
(5.29)\qquad \lesssim\ & C_\varepsilon\, N^\varepsilon \Big[\frac{L^{p-1}}{s} + L^{\frac{p}{2}} + \frac{L^{p-\frac{3}{2}}}{\sqrt{s}} \Big].
\end{aligned}$$

Recall from Lemma 5.5 that the desired bound is $\frac{L^{p-1}}{s} + L^{\frac{p}{2}}$. If $p=3$, then (5.29) does indeed agree with this bound. Since the hypotheses involving the RMP hold in case $p=3$, we are done with that case, regardless of the relative size of L and s. Let us assume now that $L\ge s$. Then (5.29) agrees with the desired bound for all p. This means that we can run the same type of inductive argument as in Corollary 4.14. We leave it to the reader to check that this proves (5.24) for all $p\ge 2$ provided $L\ge s$. Finally, if $L<s$, then $L<s\le a\le \frac{N}{L}$ and thus $L\le \sqrt{N}$. In particular, $\#\mathcal{S}\le\sqrt{N}$ in that case. In analogy with the random subset case, this suggests that $\mathcal{S}(\omega)$ are $\Lambda(p)$-sets for $2\le p\le 4$ with high probability. Although perturbed arithmetic progressions are not covered by [B1], it turns

out that the strategy from [B1] and [B2] is still relevant. More precisely, suppose first that $2 \le p \le 3$. Then (5.28) holds, even without the N^ε-term. By Plancherel, but without appealing to any RMP,

$$\mathbb{E}_{\omega_2} \sup_{x\in\mathcal{E}(\omega_2)} |x| \le \mathbb{E}_{\omega_2} \sup_{|a_n|\le 1} \Big\| \Big| \sum_{n\in\mathcal{S}(\omega)} a_n e(n\theta)\Big|^{p-1} \Big\|_2 \lesssim K_p^{\frac{p}{2}} L^{\frac{p-2}{2}}. \tag{5.30}$$

Here

$$K_p^p := \mathbb{E}_{\omega} \sup_{|a_n|\le 1} \int_0^1 \Big| \sum_{n\in\mathcal{S}(\omega)} a_n\, e(n\theta)\Big|^p \, d\theta.$$

To pass to (5.30), one writes $2(p-1) = p + (p-2)$ and then estimates the $(p-2)$-power in L^∞. Secondly, to bound the entropy term, set $q = \frac{2}{3-p}$. Then by Plancherel the distance between any two elements in $\mathcal{E}(\omega_2)$ is at most

$$\begin{aligned} \big\| g|g|^{p-2} - h|h|^{p-2} \big\|_2 &\lesssim \|g-h\|_q \big(\|g\|_2^{p-2} + \|h\|_2^{p-2} \big) \\ &\lesssim L^{\frac{p-2}{2}} \, \|g-h\|_q, \end{aligned}$$

where $g, h \in \sqrt{L}\mathcal{P}_{\mathcal{S}(\omega_2)}$, see (4.55). As before, the entropy estimate therefore reads

$$\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)}\, dt \lesssim \sqrt{L}\, L^{\frac{p-2}{2}} \sqrt{L} = L^{\frac{p}{2}}.$$

Inserting these bounds into (5.28) yields

$$K_p^p \lesssim L^{\frac{p}{2}} + (1+\sqrt{L/s}) K_p^{\frac{p}{2}} L^{\frac{p-2}{2}} \le C L^{\frac{p}{2}} + \frac{1}{2} K_p^p + C(1+L/s) L^{p-2}.$$

Since $L^{p-2} \le L^{\frac{p}{2}}$ in view of $p \le 4$, one obtains the desired bound

$$K_p^p \lesssim L^{\frac{p}{2}} + \frac{L^{p-1}}{s}$$

if $2 \le p \le 3$ and regardless of the relative size of L and s. If $3 \le p \le 4$, then the previous argument needs to be modified in two places: Firstly, there is the issue of the O-term in (5.25). However, we just showed that the RMP holds at $p-1 \le 3$, and therefore (5.26) applies here as well (even without the N^ε-term). Secondly, the entropy bounds need to be modified. In case $3 \le p \le 4$, one has $2(p-2) \le p$. Hence

$$\begin{aligned} \|g|g|^{p-2} - h|h|^{p-2}\|_2 &\le \|g-h\|_\infty \big(\|g\|_{2(p-2)}^{p-2} + \|h\|_{2(p-2)}^{p-2} \big) \\ &\le C_\varepsilon N^\varepsilon \|g-h\|_q \big(\|g\|_p^{p-2} + \|h\|_p^{p-2} \big) \\ &\le C_\varepsilon N^\varepsilon \sup_{|a_n|\le 1} \Big\| \sum_{n=1}^N a_n \xi_n(\omega_2) e(n\cdot) \Big\|_p^{p-2} \|g-h\|_q, \end{aligned}$$

with g, h as above. By the usual arguments, cf. (4.64), it follows that

$$\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)}\, dt \lesssim L\, K_p^{\frac{p-2}{2}} \le \frac{1}{2} K_p^p + L^{\frac{p}{2}}.$$

Inserting these bounds into (5.28) implies the desired bound. □

Remark 5.7. It is possible that one can make improvements on Theorem 5.6 similar to those in Theorem 4.6, thus removing the condition $L \geq s$ in some range of $p \geq 4$. This would require working with $\Lambda(p)$ type arguments as we just did in the end of the previous proof. But we do not pursue that issue here.

References

[B1] Bourgain, J. Bounded orthogonal systems and the $\Lambda(p)$-set problem. *Acta Math.* 162 (1989), no. 3-4, 227–245.

[B2] Bourgain, J. $\Lambda(p)$-sets in analysis: results, problems and related aspects. Handbook of the geometry of Banach spaces, Vol. I, 195–232, North-Holland, Amsterdam, 2001.

[B3] Bourgain, J. On $\Lambda(p)$-subsets of squares. *Israel J. Math.* 67 (1989), no. 3, 291–311.

[B4] Bourgain, J. Remarks on Halasz-Montgomery type inequalities. *Geometric aspects of functional analysis* (Israel, 1992-1994), 25-39, Oper. Theory Adv. Appl. 77, Birkhäuser, Basel 1995.

[BLM] Bourgain, J. Lindenstrauss, J. and Milman, V. Approximation of zonoids by zonotopes. *Acta Math.* 162 (1989), no. 1-2, 73–141.

[GR] Green B.; Ruzsa I. Z. On the Hardy-Littlewood majorant problem *Math. Proc. Camb. Phil. Soc.* 137 (2004), no 3, 511– 517.

[LT] Ledoux, M. and Talagrand, M. *Probability in Banach spaces. Isoperimetry and processes.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 23. Springer-Verlag, Berlin, 1991.

[HL] G.H. Hardy and J.E. Littlewood. Notes on the theory of series (XIX): a problem concerning majorants of Fourier series. *Quart. J. Math.*, 6:304–315, 1935.

[M] Mockenhaupt, G. *Bounds in Lebesgue spaces of oscillatory integrals.* Habilitationsschrift, Siegen, 1996. http://www-math-analysis.ku-eichstaett.de/ gerdm/.

[Mont] H. Montgomery. *Ten Lectures on the interface between Analytic Number Theory and Harmonic Analysis.* CBMS, Vol. 84. AMS, 1994.

[PT-J] Pajor, A.; Tomczak-Jaegermann, N. Subspaces of small codimension of finite-dimensional Banach spaces. *Proc. Amer. Math. Soc.* 97 (1986), no. 4, 637–642.

[P] Pisier, G. *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.

[T] Talagrand, M. Sections of smooth convex bodies via majorizing measures. *Acta Math.* 175 (1995), no. 2, 273–300.

[SZ] Salem, R. and Zygmund, A. Some properties of trigonometric series whose terms have random signs. *Acta Math.* 91 (1954), 245-301.

[Zyg] A. Zygmund, *Trigonometric Series.* Cambridge University Press, 1959.

MOCKENHAUPT: FACHGEBIET MATHEMATIK, KATHOLISCHE UNIVERSITÄT EICHSTÄTT-INGOLSTADT, OSTENSTRASSE 26, 85972 EICHSTÄTT, GERMANY.
email: **gerdm@ku-eichstaett.de**

SCHLAG: DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, 5734 SOUTH UNIVERSITY AVENUE, CHICAGO, IL 60637, U.S.A.
email: **schlag@its.caltech.edu**